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Reformulating decision theory using fuzzy set theory and Shafer's theory of evidence

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Abstract

Utilities and probabilities in decision theory are usually assessed by asking individuals to indicate their preferences between various uncertain choices. In this paper, we argue that

(1) The utility of a consequence can be assessed as the membership function of the consequence in the fuzzy set '*satisfactory*'.

(2) The probability of an event, instead of being directly assessed, should be inferred from the *evidence* associated with that event. The degree of evidence is quantified using Shaferian basic probability assignments.

In addition, we use the Heisenberg Uncertainty Principle to argue for a change in one of the technical assumptions underlying decision theory. As a result of this change, some kinds of evidence will be observable in certain experiments but unobservable in others. Since probabilities are defined over the potential outcomes of an experiment, they will only be defined over some, but not all, the evidence. As a result, the probabilities associated with different experiments could be inconsistent.

This formulation emphasizes the importance of new distinctions (and not just new information) in updating probabilities.

We argue that this formulation addresses many of the observed empirical deviations between decision theory and experiment. It also addresses the anomalies of quantum physics. We close with a brief discussion of directions for further research.

Keywords: Decision analysis; Possibility theory; Non-additive measure; Psychometry and measurement; Quantum physics

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0. Introduction

Savage [45] showed that a rational individual can be modeled as making decisions by

(1) Assigning a utility $u(x)$ to each possible consequence x.
(2) Assigning a probability $p(s)$ to each possible state of nature, s.
(3) Defining the utility of a decision d leading to consequence $d(s)$ in state of nature s as

$$Eu(d) = \sum_x u(d(s))\,p(s).$$

(4) Choosing that decision d with the maximum value of $Eu(d)$.

Unfortunately, Savage's theory has been found to be inconsistent, in many ways, with how people actually behave. In addition, his theory of probability appears to be inconsistent with the theory of probability used in quantum physics. Some have dismissed these findings and argued that Savage's theory is normative and the evidence reflects irrational behavior. Others have argued for modifications in Savage's notions of rationality.

This paper presents a third approach which involves reformulating Savage's notions of probability and utility using both fuzzy set theory and Dempster–Shafer theory. This reformulation will make the theory much more consistent with empirical evidence in cognitive psychology, organizational behavior and quantum physics.

The first section focuses on the notion of utility in Savage's theory. There have been many proposals for developing a fuzzy theory of decision-making by fuzzifying utility [12,41,44,39]. But instead of fuzzifying utility, this paper proposes replacing it with a mathematical equivalent notion. Specifically, we show that the utility of any consequence can always be reinterpreted as the fuzzy set membership of that consequence in the fuzzy set 'satisfactory'.

The second and third sections focus on the notion of probability in Savage's theory. Dubois et al. [13] had proposed modifying several of Savage's axioms to develop a fuzzy counterpart of Savage's theory. Like most authors, they focused on Savage's substantive axioms which are central to the framework of rationality which Savage was constructing. Their paper noted, in passing, that

> *The two axioms, Savage's* 6^{th} *and Savage's* 7^{th}*, are clearly technical, not so necessary as the other ones and not so essential to the framework (p. 466).*

This paper will not make any changes in any of Savage's substantive axioms. But as our second section notes, Savage's 6th axioms contradicts the Heisenberg Uncertainty Principle and hence requires some modification. As we will show, this modification provides a way of accommodating not only the celebrated anomalies of quantum physics but also many of the empirical anomalies associated with human decision-making.

Given this modification of Savage's 6th axiom, our third section extends arguments in quantum logic to develop a theory of multiple probabilities for different experiments. In this theory, there is a single frame of discernment describing all the possible outcomes of several possible experiments. Shafer's basic probability assignments are used to measure the degree of evidence associated with the various states in this frame of discernment. The probabilities for outcomes observable in an experiment are derived from the evidence associated with those outcomes. (Thus while it is common to directly assess probabilities, our approach advocates directly assessing evidence weights and using them to calculate probabilities.) Since different experiments can observe different states (and thus different evidence), the probabilities associated with different experiments may be inconsistent.

As we will show, reformulating expected utility theory on the basis of fuzzy set theory and Shafer's theory of evidence makes it considerably more consistent with empirical evidence in cognitive psychology and organizational behavior. In addition, it makes the theory consistent with probabilities in quantum physics which, for the most part, have been considered outside the scope of decision theory.

1. Utility as a fuzzy membership function

We review Simon's theory of bounded rationality as well as a proposed new interpretation of utility. We then show that utility can be interpreted as a fuzzy set membership function.

1.1. Simon's theory of bounded rationality

In 1954, Savage enunciated his axioms of rational behavior which implied that a rational individual implicitly maximizes expected utility. A year later, Simon [51] enunciated his theory of bounded rationality because of his concern that utility theory presumed "*a well-organized & stable system of preferences and a skill in computation*" that was unrealistic in many decision contexts. In Simon's theory, individuals simply looked for the first decision alternative that was '*satisfactory*', i.e., that met some prespecified target. Over the past 50 years, both theories have been subjected to many empirical tests. In his Nobel Prize Lecture (1978), Simon [52] concluded that utility theory had been refuted and declared

> *There can be no doubt that...the assumptions of perfect rationality...do not even remotely describe the processes that human beings use for making decisions in complex situations.*

He also asserted that his own theory had been confirmed:

> *Research in information processing psychology provides conclusive evidence that the decision-making process in problem situations conforms closely to the models of bounded rationality ...the notions of bounded rationality* [*include*] *the need to search for decision alternatives, the replacement of optimizing by targets and satisficing and mechanisms of learning and adaptation.*

See Simon [53,54] for discussions of the empirical evidence in support of bounded rationality. Simon concluded that human behavior should be modeled as satisfying instead of optimizing.

1.2. A new interpretation of utility

Simon's theory was originally enunciated for target-setting bureaucracies so that a '*satisfactory*' outcome could be defined as an outcome meeting that target. But in many other settings, targets are not precisely defined. Hence it is not clear how to define what it means to be '*satisfactory*'. For example, suppose we define a satisfactory retirement as one in which we continue to live at our current standard of living. We want to save enough money to have a satisfactory retirement.

But since future prices are uncertain, it is impossible to know exactly how much income would be required for a satisfactory retirement.

Now suppose a retirement income of a hundred thousand dollars a year would definitely ensure a satisfactory retirement. Also suppose that a retirement income of only $20,000 a year would definitely ensure an unsatisfactory retirement. If we used Von Neumann and Morgenstern's procedure for assigning utilities to various amounts of money, we would treat a hundred thousand dollars a year as our best outcome and assign it a utility of one. Likewise we would treat twenty thousand dollars a year as our worst outcome and assigned it a utility of zero. We would assign a utility of $u(x)$ to an intermediate income of x if we are indifferent between getting x versus getting a lottery offering a $u(x)$ chance of the best outcome and a $(1-u(x))$ chance of the worst outcome. Since the best outcome ensures a satisfactory retirement and the worst outcome guarantees an unsatisfactory retirement, this lottery is equivalent to a $u(x)$ chance of a satisfactory retirement and a $(1-u(x))$ chance of an unsatisfactory retirement.

Hence assigning a utility of $u(x)$ to x implies that we are indifferent between x and a lottery offering us a $u(x)$ chance of a satisfactory retirement and a $(1-u(x))$ chance of an unsatisfactory retirement. In other words, $u(x)$ must equal the probability of x leading to a satisfactory retirement. If T is a random variable describing the unknown amount of money required for a satisfactory retirement, then $u(x)=Pr(x\geqslant T)$. Now consider a lottery X which gives a payoff of x with probability $P(X=x)$. If X and T are independent, the expected utility of the gamble is

$$Eu(X)=\sum_x u(x)P(X=x)=\sum_x P(x\geqslant T)P(X=x)=P(X\geqslant T).$$

Hence, the expected utility of a gamble is the probability the gamble leads to a payoff that exceeds what is required for a satisfactory retirement. For lotteries with monetary payoffs, Castagnoli and LiCalzi [8] proved a general equivalence between expected utility with Von Neumann/Morgenstern utilities and the probability of exceeding a random threshold, T.

Bordley et al. [7] extended that argument to Savage's model with potentially non-monetary consequences, c. Consider a specific consequence c and suppose $p(c)$ is the probability that consequence c will meet the uncertain criteria for being satisfactory. For any decision d, define a random variable X_d such that the probability of X_d equaling consequence c is the probability that a state s occurs for which $d(s)=c$. Then the probability of decision d leading to a 'satisfactory' outcome is just

$$\sum_s P(X_d=c)p(c).$$

Now the Savage axioms imply the existence of a probability function, P, and a utility function, u, such that the utility of a decision can be written as

$$\sum_s P(s)u(d(s))=\sum_c P(X_d=c)u(c).$$

Since Savage utilities must be bounded between zero and one, it is always possible to interpret $u(c)$ as $p(c)$, the probability of the consequence being '*satisfactory*'. Hence, the utility of decision d is the probability the decision leads to a consequence which satisfies the uncertain requirements for being '*satisfactory*'.

1.3. Interpretation in terms of fuzzy membership functions

What does this result show? Savage's axioms of rational choice imply

Interpretation 1. *The individual acts 'as if' there exists a probability P and a utility function, u, such that the individual prefers d to d* if and only if* $Eu(d) > Eu(d^*)$.

But the Castagnoli, Bordley and LiCalzi results show that Savage's axioms also imply

Interpretation 2. *The individual acts 'as if' there were a probability P and uncertain requirements for being satisfactory such that the individual prefers d to d* if the probability of d yielding a consequence meeting these uncertain requirements exceeds the probability of d* yielding a consequence meeting these uncertain requirements.*

In this formulation, a consequence falls in the set '*satisfactory*' if it meets the uncertain requirements for being satisfactory. This makes '*satisfactory*' a random set [38] since the criteria determining membership in the set of satisfactory outcomes is random. Hence, the utility of a consequence is the probability that the consequence is a member of the random set '*satisfactory*'.

But it has been shown that the probability of belonging in a random set can equivalently be interpreted as the membership function in a fuzzy set [40,5,35]. Hence, the expected utility of a consequence is its expected membership in the fuzzy set '*satisfactory*'. Thus Savage's axioms also imply

Interpretation 3. *The individual chooses that decision with the maximum expected membership in the fuzzy set 'satisfactory'.*

As a result, Savage's axioms, which are customarily interpreted as antithetical to fuzzy set theory, can potentially be reinterpreted as implying fuzzy set theory for preferences.

2. The limits of probability theory

2.1. Borel fields in Kolmogorov and Savage's probability theory

Kolmogorov's theory of probability [33] postulates an underlying sample space S whose elements are referred to as states. Following Savage, we think of a 'state' as "*a description of the world, leaving no relevant aspect undescribed*". Note that states need not be observable. Kolmogorov then postulated a set, B, consisting of some, but not necessarily all, subsets of S. This set is a Borel-field, i.e., the complement of any event in B lies in B and arbitrary countable unions of any collection of events in B lie in B. The elements of B are called events and are interpreted as the observable outcomes of some hypothetical (or actual) experiment. Because B need not include all possible subsets of S, not every state will be an event.

Probabilities are only defined over the events in B and are not defined over events not in B. Hence some states in S may not be assigned probabilities. Given these definitions, a Kolmogorov probability space is defined by the three elements $\{S, B, P\}$. If an event E in B is a disjoint union of events $A_1 \ldots A_n$ in B, then

$$P(E) = \sum_k P(A_k).$$

Dynkin [15] and Fine [17] discuss various relaxations of Kolmogorov's requirements.

Now Savage's theory of rational choice, which leads to a theory of probability, presumes the following axiom P6:

Suppose that act g is strictly preferred to act h. Then for every f, there is a finite partition of S such that if g' agrees with g and h' agrees with act h except on an arbitrary element of the partition, g' and h' being equal to f there, then h will be strictly preferred to g' and h' will be strictly preferred to g'.

This axiom implies that probabilities are defined over all possible subsets of S, i.e., B must be the powerset of S. Savage recognized that his formulation imposed a restriction on Kolmogorov's mathematical rules of probability and wrote:

It is not usual to suppose, as has been done here, that all sets have a numerical probability, but rather that a sufficiently rich class of events do so, the remainder being considered unmeasurable… . But the theory being developed here does assume that probability is defined for all events, that is for all sets of states (p. 40).

Shafer [49, p. 119] would later criticize Savage's deviation from Kolmogorov:

In the past, many students of probable reasoning have sought to establish a fixed framework for their speculations by postulating the existence of an ultimate detailed set of 'possible states of nature'—a frame of discernment so fine that it encompasses all possible distinctions and admits of no further refinement…we must reject the postulation of such an ultimate refinement. This rejection is compelled by the purely epistemic nature of the role played by the frame of discernment… . It cannot embody concepts and distinctions that one has never heard of… . Hence a realistic theory of evidence will deal with frames that do not even encompass all the knowledge we do have and will explicitly allow for their refinement.

In this section, we shall argue that there is extensive empirical evidence supporting Shafer's rejection of Savage's assumption. While Lindley (1974) has argued that empirical evidence cannot refute Savage's axioms, we will show that axiom P6 makes certain implicit empirical assumptions that have been solidly refuted by quantum physics. This is consistent with Shafer's more general observation [50] that

today's Bayesian statisticians often contend that empirical facts are completely irrelevant to the normative interpretation. People should obey Savage's postulates, and what they actually do has no relevance to this imperative (Lindley, 1974*). I shall argue that this is wrong. The normative interpretation cannot be so thoroughly insulated from empirical facts. Savage's argument for the normativeness of his postulates cannot be made without assumptions that have empirical context,*

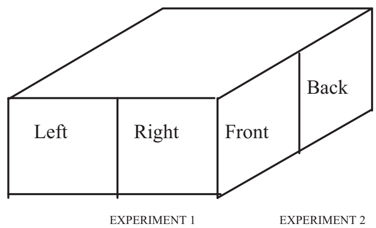
2.2. Quantum physics and the Savage axioms

According to Savage's axioms, an event consisting of exact specifications of an electron's position can always be subdivided into events including exact specifications of the electron's position and exact specifications of its momentum. But the Heisenberg Uncertainty Principle indicates that it's simply wrong to even think about electrons in this way. In the *Journal of Mathematical Physics*, Foulis and Randall [20] wrote:

> *The grand canonical measurement of classical mechanics…permits one to determine simultaneously the location and the momentum of all the particles of a physical system in quantum mechanics, the celebrated Heisenberg commutation rules reject both determinism and even an in principle possibility of a grand canonical measurement. Thus in quantum mechanics, we are denied the convenience of a single classical sample space in terms of which we are always able to confirm or refute the measurement of an observable.*

To understand how this impacts probability theory, we turned to a thought experiment [19] from quantum logic. Imagine a firefly in a box with a window to the south with two panes labeled left and right and a window to the east with two panes labeled front and back.

THE FIREFLY BOX



Experiment 1 involves observing the firefly through the south window. If you perform experiment 1, you either observe the firefly to be on the right side of the box (event *Right*), on the left side (event *Left*) or will not observe anything (event n) if the firefly's light is not on. Hence the Borel-field for this experiment consists of

$$\{\emptyset\ Right, Left, n,\ (Right\ or\ Left),\ (Right\ or\ n),\ (Left\ or\ n),\ (Right\ or\ Left\ or\ n)\}$$

Experiment 2 corresponds to observing the firefly through the east window. If you perform experiment 2, you will either observe the firefly to be on the front side of the box (event *Front*), on the back (event *Back*) or will not observe anything (event n) if the firefly's light is not on. Hence, the Borel-field for the second experiment is

$$\{\emptyset, Front, Back, n,\ (Front\ or\ Back),\ (Front\ or\ n),\ (Back\ or\ n),\ (Front\ or\ Back\ or\ n)\}$$

Suppose that looking through a window scares the firefly and causes it to change its position. Hence, the result of doing experiment 2, after we have done experiment 1, will differ from the result of doing experiment 2, without having done experiment 1. As a result, the event (*Left and Back*) is not observable because *Left* is only observable by the first experiment and *Back* is only observable by the second experiment and it's impossible to perform both experiments on the same system. On the other hand, there are some events, i.e., event '*n*', that are common to both Borel-fields. As a result, the event '*Left or Right*' will be true whenever '*Front or Back*' is true.

Birkhoff and Von Neumann [2] showed that this simple situation leads to a violation of the distributive law of conventional logic. [1] Their work created a new field, quantum logic, which presumes that there is a set (or manual) of multiple Borel-fields, each associated with different probabilities. [2] This theory generalizes Kolmogorov's theory by allowing for multiple Borel-fields. However, it refutes Savage's presumption of a unique Borel-field associated with the state-space S.

Savage recognized that his proposed modification of Kolmogorov was controversial and discussed how his theory would change if he followed Kolmogorov and defined probabilities over Borel-fields. He wrote:

> *If one is unwilling to insist on comparisons between every pair of states or events, then in the same spirit, it is inappropriate to insist on comparisons between every pair of acts. All that has been, or is to be, formally deduced in this book concerning preferences among sets, could be modified, mutatis mutandis, so that the class of events would not be the class of all subsets of S, but rather a Borel field, that is, a sigma-algebra… . Indeed the whole theory could be done for abstract sigma-algebras without reference to sets at all, and this might have some actual advantage, since it would make possible the identification of events with propositions in almost any formal language, even one unable to formulate at all the complete descriptions I call states (p. 42).*

In other words, if we modify axiom (P6) to make it consistent with Kolmogorov, then we can still derive expected utility theory although expected utility can now only be used to compare gambles defined over the same Borel-field. The next section presents empirical evidence indicating that such a restricted utility theory would be considerably more consistent with empirical evidence than Savage's original version of utility theory.

2.3. The Allais Paradox and the Savage axioms

We start with the Allais Paradox. An individual is presented with two sets of choices:

(1) A choice between
- Gamble 1, a million dollars or
- Gamble 2, an 89% chance of a million dollars, a 10% chance of \$2.5 million dollars and a 1% chance of nothing.

(2) A choice between
- Gamble 3, an 11% chance of a million and an 89% chance of nothing or
- Gamble 4, a 90% chance of nothing and a 10% chance of \$2.5 million.

[1] Suppose gamble A pays off if the firefly is either in the *Left* or *Right* of the box as well as being in the *Front* of the box, while gamble B pays off if the firefly is either in the *Left* and *Front* of the box or if it is in the *Right* and *Front* of the box. The distributive law of logic indicates makes both gambles equivalent. But the first gamble pays off if the firefly is observed in the front of the box, while the second gamble can never be resolved. Hence, the distributive law of logic fails.

[2] For more information, see the web-site for the *International Quantum Structures Association.*

Many people chose gamble 1, the sure thing over gamble 2 and chose gamble 4 over gamble 3. To show that these choices were inconsistent, Savage defined events E^1, E^2 and E^3 with probabilities 10%, 1% and 89% and showed that the gambles could be defined over these events as follows:

	E^1	E^2	E^2
Gamble 1	$1 Million	$1 Million	$1 Million
Gamble 2	$2.5 Million	$0 Million	$1 Million
Gamble 3	$1 Million	$1 Million	$0 Million
Gamble 4	$2.5 Million	$0 Million	$0 Million

Given this formulation, it is clear that an individual who prefers gamble 1–2 should similarly prefer 3–4, in violation of observed behavior.

Now Savage's 'proof' that the Allais anomalies are irrational presumes that all four gambles are analyzed using the same three events: E^1, E^2 and E^3. In our view, this is equivalent to presuming that all four gambles can be defined over the same Borel-field. But it seems reasonable to assume that the original gambles were actually defined over the following different Borel-fields, i.e.,

Gamble 1's Borel-field is defined by a single event, the union of E^1, E^2 and E^3.
Gamble 2's Borel-field is defined by all three events, E^1, E^2 and E^3.
Gamble 3's Borel-field is defined by E^3 and its complement.
Gamble 4's Borel-field is defined by E^1 and its complement.

If we follow Kolmogorov in assuming that the rules of probability only constrain gambles defined over the same Borel-field, then the probabilities assessed for these four gambles need not be consistent. Since applying expected utility to describe choices between the gambles presumes the same underlying set of probabilities, expected utility will not describe comparisons of these gambles. And this, of course, is what is observed in the Allais experiments.

Hence rejecting Savage's restriction of Kolmogorov—which amounts to allowing gambles to be defined over different Borel-fields—means that the 'anomalous' Allais Paradox behavior is irrational, not because individuals violated utility theory, but because utility theory was used to compare gambles defined over different Borel-fields.

Work by Harless and Camerer (1994) suggests that this same explanation resolves many (perhaps most) of the observed discrepancies between expected utility theory and observed individual choices. Harless and Camerer reviewed the experimental evidence on a wide range of theories of rational choice, including expected utility theory, prospect theory and other formalisms. They found that the critical factor determining whether expected utility described preferences between two gambles was whether or not the two gambles being compared were defined over the same uncertainties (i.e., whether the gambles had the same support.) As they wrote

> *Analysis of* 23 *datasets, using several thousand choices, suggests a menu of theories which sacrifice the least parsimony for the biggest improvement in fit. The menu is mixed fanning, prospect theory, expected utility and expected value. Which theories are best is highly sensitive to whether gambles in a pair have the same support* (*EU fits better*) *or not* (*EU fits poorly*).

Since gambles constructed on different Borel-fields will necessarily have different supports, the Harless and Camerer evidence could be interpreted as suggesting that utility theory fails when used to compare gambles assessed across different Borel-fields.

2.4. The Dutch book argument

One of the standard arguments against allowing an individual to have multiple sets of probabilities is DeFinetti's Dutch book argument. DeFinetti argued that if an individual assigned an inconsistent set of probabilities over a set of events, then it would be possible to structure a set of gambles over these events which would lead the individual to make choices guaranteed to cost him money. Applying the Dutch book argument to our context involves defining a set of gambles across multiple Borel-fields and presuming that the individual will then evaluate these gambles using the multiple utility functions defined for these different Borel-fields. However, our formulation only allows the individual to use utilities to compare gambles defined over the same field of events. It does not allow the individual to use utility functions to compare a gamble defined over one Borel-field with a gamble defined over a different Borel-field. (In the firefly experiment, this would be equivalent to defining a gamble which can never be resolved because the outcomes of one gamble are only observable by the first experiment while the outcomes of the second gamble are only observable by the second experiment.) Hence, our individual would refuse to make certain comparisons and would thus avoid the Dutch book.

Thus our formulation is logically consistent and is not subject to manipulation by the Dutch book.

3. A new form of probability theory

3.1. Shafer's basic probability assignments

Given our proposed generalization of Savage, we can characterize Savage's theory (like Kolmogorov's theory) by a state-space, S, a Borel-field, B, and a probability function. As we discussed, quantum logic replaces Kolmogorov's single Borel-field, B, by a family (or manual) of Borel-fields, M, each with their own associated probabilities. This raises the question of how the probabilities associated with one Borel-field, B, relate to the probabilities associated with a different Borel-field, B'. This is currently an unsolved problem in quantum logic. In this paper, we propose solving this problem using Shafer's theory of evidence.

We first suppose that the state space, S, is atomic (e.g., $S=(1,2,3,4)$). (We will refer to (1)–(4), which are the atoms of the powerset of S, as singletons.) As a result, any Borel-field, B, generated from S (e.g., $\{\emptyset,(12),(34),(1,2,3,4)\}$) will also be atomic although the atoms of the Borel-field (which are (12) and (34) in this case) need not be atoms of the state space. It will be helpful to define the basis of $B, S(B)$, as the set of atoms in Borel-field B. (Note that the Borel-field, B, is just the powerset of $S(B)$.)

We could think of Kolmogorov's theory as first assigning probabilities to the atoms of the Borel-field (i.e., to the elements of $S(B)$) and then assigning probabilities to non-atomic events using the rule:

Definition 1. For a non-atomic event E in Borel-field B defined over S, the probability of event E is given by

$$P(E) = \sum_{A \text{ in } E} P(A).$$

This definition ensures that probabilities are additive over the Borel-field.

Note that Kolmogorov's theory does not specify how probabilities are assigned to atomic events in the Borel-field. To specify these probabilities, recall that Shafer [49] attached non-negative *basic probability assignments*, m, to all the elements of the powerset of S and used these basic probability assignments to generate belief functions. (As he noted, under special conditions, the belief function corresponds to a probability.) We now propose using Shafer's basic probability assignments to determine probabilities for atomic events in Borel-fields as follows:

Definition 2. For an atomic event A in Borel-field B over state-space S, the probability of event A is given by:

$$P(A) = m(A) \Bigg/ \sum_{A \text{ in } S(B)} m(A) \,.$$

Definition 2 is formally similar to the standard definition of probabilities in terms of measure functions. But instead of using the conventional additive measures, we use Shafer's basic probability assignments. [3]

To illustrate the implications of this definition, we return to the 'firefly in a box' example. In the firefly problem, the state-space S consisted of

$$S = \{Right \,\&\, Front,\ Right \,\&\, Back,\ Left \,\&\, Front,\ Left \,\&\, Back,\ n\}$$

The basic probability assignments for the powerset of S are

$$\{m(Left \,\&\, Front),\ m(Left \,\&\, Back),\ m(Right \,\&\, Front),\ m(Right \,\&\, Back),\ m(n),$$
$$m(Left),\ m(Front),\ m(Left \,\&\, Front \text{ or } Right \,\&\, Back),\ m(Left \,\&\, Back \text{ or } Right \,\&$$
$$Front),\ m(Back),\ m(Right),\ m(Left \text{ or } Right \,\&\, Front),\ m(Left \text{ or } Right \,\&\, Back),\ m$$
$$(Front \text{ or } Right \,\&\, Back),\ m(Left \,\&\, Back \text{ or } Right),\ m(Left \text{ or } Right),$$
$$m(Left \text{ or } Right \text{ or } n)\}$$

The firefly problem involves two different experiments and thus two different Borel-fields. The atomic events for the first Borel-field (i.e., the basis of the first experiment) are $\{Left, Right, n\}$. Given

[3] Suppose that A and A^* are atoms of experiment 1 and are also atoms of experiment 2. Then our formulation implies that $P(A)/P(A^*)$, i.e., the relative probability of the two events, will be the same for both experiments.

definition 2, the probabilities for these events are

$$P(Left) = m(Left)/[m(Left) + m(Right) + m(n)]$$
$$P(Right) = m(Right)/[m(Left) + m(Right) + m(n)]$$
$$P(n) = m(n)/[m(Left) + m(Right) + m(n)]$$

We then use Definition 1 to generate probabilities for all other events in that Borel-field. Thus $P(Right\ or\ n) = P(Right) + P(n)$.

The atomic events for the second Borel-field are $\{Front, Back, n\}$ and the probabilities for these atomic events, given Definition 2, are

$$P(Front) = m(Front)/[m(Front) + m(Back) + m(n)]$$
$$P(Back) = m(Back)/[m(Front) + m(Back) + m(n)]$$
$$P(n) = m(n)/[m(Front) + m(Back) + m(n)].$$

We then use Definition 1 to generate probabilities for all other events in that Borel-field.

Note that neither set of probabilities depend on $m(Front\ \&\ Left)$ while both depend on $m(n)$. Thus probabilities, unlike Shafer's belief function, depend on some but not all of the basic probability assignments. This formulation involves replacing Savage's probability space $\{S, P\}$ and Kolmogorov's probability space $\{S, B, P\}$ with a new kind of probability space $\{S, m, M\}$ where m are Shaferian basic probability assignments over the powerset of S and M is a set of Borel-fields constructed from S. We now show that this formulation describes uncertainty in both quantum mechanics and cognitive psychology.

3.2. Applying the formalism to uncertainty in quantum physics

Because of the Heisenberg Uncertainty Principle, probability theory is not used to model quantum mechanical uncertainty. Instead, it is replaced by probability amplitude mechanics [16] which defines a complex wave function, $\psi(\mathrm{A})$, for every state A in the set of all possible states, S. We can write this complex wave function as

$$\psi(A) = (P(A))^{1/2}\ \exp((-1)^{1/2}\theta(A)),$$

where $\theta(A)$ is a new factor reflecting the phase of the wave. At this point, these states are not necessarily observable events because they have been defined independent of any experiment.

Now suppose state A is observable in an experiment called the refined experiment. Then probability amplitude mechanics writes the probability of A in the refined experiment as proportional to the complex square of $\psi(A)$. Also, suppose that A is composed of the two disjoint events A^* and A^{**}. Suppose that the experiment can also discriminate between A^* and A^{**}. Then

Empirical observation 1. *The probability of observing A^* or A^{**} in the refined experiment is proportional to $|\psi(A^*)|^2 + |\psi(A^{**})|^2$ which is proportional to $P(A^*) + P(A^{**})$.*

Now consider a different experiment (called the coarse experiment) which can observe A^* or A^{**} but cannot separately discriminate between A^* and A^{**}. Then

Empirical observation 2. *The probability of observing A^* or A^{**} in the coarse experiment is proportional to $|\psi(A^*)+\psi(A^{**})|^2$ which is proportional to $P(A^*)+P(A^{**})+2(P(A^*)P(A^{**}))^{1/2}\cos(\theta(A^*)-\theta(A^{**}))$.*

This last term is commonly referred to as an interference term and can cause the probability of A^* or A^{**} to vary from zero (which is called complete destructive interference) to $2(P(A^*)+P(A^{**}))$. As Feynmann noted, the disparity between empirical observation 1 and empirical observation 2 is

> *a phenomenon which is impossible, absolutely impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery…in telling you how it works, we will have told you about the basic peculiarities of all quantum mechanics*

As we now show, probability amplitude mechanics is encompassed with our proposed formulation. First we attach basic probability assignments as follows:

(Q1) If A is an atomic event in the powerset of S (i.e., is a singleton), then

$$m(A)=|\psi(A)|^2.$$

(Q2) If E is an element of the powerset of S consisting of singletons A in E, then

$$m(E)=\left|\sum_{A \text{ in } E}\psi(A)\right|^2.$$

The basic probability assignments will always be nonnegative with $m(E)$ varying from 0 to $2\sum_{A \text{ in } E} m(A)$. Since each wave function is determined by two values (a real number and an imaginary number), all the basic probability assignments are determined by $2|S|$ numbers.

In the refined experiment, A^* and A^{**} were atoms of the Borel-field. Hence their probability was determined by Definition 1. As a result $P(A^*)$ was proportional to $m(A^*)$ and $P(A^{**})$ was proportional to $m(A^{**})$. The probability of the non-atomic event A^* or A^{**} was determined by Definition 2 and is proportional to $m(A^*)+m(A^{**})$ which agrees with empirical observation 1.

In the coarse experiment, A^* or A^{**} was an atom of the Borel-field (since A^* was not an observable atom of the field and A^{**} was not an observable atom of the field.) Hence its probability was determined by Definition 1. As a result $P(A^*$ or $A^{**})$ was proportional to $m(A^*$ or $A^{**})$ which agrees with empirical observation 2.

Thus probability amplitude mechanics is a special case of our Shaferian formulation with the basic probability assignments given by (Q1) and (Q2).

3.3. Applying the formalism to uncertainty in cognitive psychology

In cognitive psychology, Fischoff et al. [18] seminal work on fault trees demonstrated an important bias in individual assessments of probability. Thus, consider the following two experiments from

Tversky and Koehler [56]:

(C) In experiment, E_C, Stanford undergraduates are asked to assess the frequency of death from natural causes. Their average estimate is 58%.
(R) In experiment, E_R, Stanford undergraduates are asked to separately assess the frequency of death from heart disease, the frequency of death from cancer and the frequency of death from natural causes other than heart disease or cancer. When their estimates of the frequency of death from these various specific natural causes were summed, the average estimate was 73%.

Hence probability estimates associated with a more detailed specification of events (E_R) were generally higher than the probability estimates associated with a more 'high-level' specification of events (E_C). This phenomenon has been replicated in many other contexts [18,21,28,42,43,55,57].

To model this, Tversky and Koehler [56] proposed assigning a support function $s(A)$ to every state A in S. They also estimate a parameter $k<1$. Now consider a refined experiment in which the probability of events A^* and A^{**} were both assessed. (In this case, they referred to A^* or A^{**} as an *explicit* disjunction because it was a disjunction of events treated in the experiment.) They postulated that:

Empirical hypothesis 1. *The probability of the explicit disjunction*, $P(A^*$ *or* $A^{**})$, *is proportional to* $s^k(A^*)+s^k(A^{**})$.

Now consider a second coarse experiment in which the subjects do not assess probabilities for either A^* or A^{**} but are asked to assess a probability for A^* or A^{**}. They referred to A^* or A^{**} as an implicit disjunction. They postulated that

Empirical hypothesis 2. *The probability of the implicit disjunction*, $P(A^*$ *or* $A^{**})$, is proportional to $(s(A^*)+s(A^{**}))^k$.

Since $k<1$, we have

$$s^k(A^*)+s^k(A^{**}) > (s(A^*)+s(A^{**}))^k$$

so that the probability assessed for an explicit disjunction was higher than the probability for an implicit disjunction. As a result, their theory, which they called support theory, reproduces the observed subadditivity in individual probability assessments. They tested this model in several experiments and found it consistent with empirical evidence.

As we now show, support theory is encompassed within our proposed formulation. First we attach basic probability assignments as follows:

(C1) If A is an atomic event in the powerset of S (i.e., a singleton), then

$$m(A)=s^k(A).$$

(C2) If E is an element of the powerset of S consisting of atoms A in E, then

$$m(E)=\left(\sum_{A \text{ in } E} s(A)\right)^k.$$

Now that only $|S|+1$ values need to be specified in order to specify all the basic probability assignments.

In an experiment in which A^* and A^{**} are both observed (i.e., are atoms of the Borel-field), the probability of A^* or A^{**} is given by Definition 1 and hence is proportional to $s^k(A^*)+s^k(A^{**})$. In an experiment in which A^* and A^{**} are not observed but A^* or A^{**} is observed, A^* or A^{**} is an atom of the Borel-field. In that case, the probability of A^* or A^{**} is given by Definition 2 and hence is proportional to $(s(A^*)+s(A^{**}))^k$.

Thus, probability amplitude mechanics is a special case of our Shaferian formulation with the basic probability assignments given by (C1) and (C2).

3.4. Restrictions on basic probability assignments

While Shafer's theory allows us $2^{|S|}$ degrees of freedom in specifying basic probability assignments, empirical theories in cognitive psychology and quantum physics only required $2|S|$ degrees of freedom. Since a theory with $2|S|$ degrees of freedom is much easier to test and apply than a theory with $2^{|S|}$ degrees of freedom, we now develop some axioms for assessing basic probability assignments which will give us a theory with $2|S|+1$ degrees of freedom that more closely corresponds to these empirical theories in quantum physics and cognitive psychology.

The simplest restriction would begin by attaching basic probability assignments to singletons and then writing the basic probability assignment for the disjunction of singletons as a function of the basic probability assignments for each singleton. But it is obvious that the basic probability assignment for the disjunction of singletons must also depend on the degree of affinity between the singleton events involved. For example, suppose A^* is the event '*the creature is a man*', A^{**} is the event '*the creature is a woman*' and A^{***} is the event '*the creature is a squid*'. Then the disjunction A^* or A^{**} corresponds to the event '*the creature is a human being*'. It is easy to imagine evidence that would allow us to conclude that the creature is a human being without telling us anything about whether it is a male human being or a female human being. Hence, $m(E)=m(A^*$ or $A^{**})$ will be positive and could potentially exceed $m(A)+m(A^*)$. On the other hand, consider 'A^* or A^{***}': the creature is a man or squid.' It is hard to conceive of any evidence supporting the truth of A^* or A^{***} which would not distinguish between the occurrence of A^* and the occurrence of A^{***}. Hence, $m(E)=m(A^*$ or $A^{***})$ will be approximately zero even though $m(A^*)$ and $m(A^{***})$ might be positive. Thus, we must consider the affinity between A and A^* in attaching a basic probability assignment to the disjunction of A and A^*.

How do we measure affinity? In econometrics, it is common to model the correlation between product sales by writing a customer's probability of choosing a product as an average of several secondary choice probabilities. (This is sometimes called the heterogeneous logit or mixed multinomial logit model [1,32]). In this paper, we will similarly model affinity between states by writing the basic probability assignment for a state as an average of two secondary basic probability assignments, m_1 and m_2.

We first attach the m_1 assignment to singleton events and then write the m_1 assignment for the disjunction of singletons as a function of the m_1 assignment of each if the singletons. We then impose the following six restrictions on how $m_1(E)$ relates to the m_1 values for the singleton events

$A_1 \cdots A_n$ which compose event E:

(1) *Monotonicity*: $m_1(E)$ increases if any of the $m_1(A_j), (j=1,\ldots,n)$ values increase.
(2) *Independence*: Suppose event E consists of a union of $A_1 \ldots A_n$ and E^* consists of a union of $A_1^* \ldots A_n^*$. Let J be a set of indices and suppose that for some a_j, $m_1(A_j)=m_1(A_j^*)=a_j$ for j in J. Suppose that given the values of $m_1(A_k)$ and $m_1(A_k^*)$ for k not in J, we have $m_1(E)>m_1(E^*)$. Then $m_1(E)>m_1(E^*)$ will still hold if we replace a_j by any different set of values a_j^*. This result holds for any index set J.
(3) *Continuity*: $m_1(E)$ is a continuous function of $m_1(A_1)\ldots m_1(A_n)$.
(4) *Restricted solvability*: Suppose $m_1(A_j)=m_1(A_j^{**})$ for j not equal to r and $m_1(A_r)>m_1(A_r^{**})$. Then, if $m_1(E)>m_1(E^*)>m_1(E^{**})$, there exists a value m_1^{**} such that changing $m_1(A_r)$ to m_1^{**} makes $m_1(A)=m_1(A^*)$.
(5) *Archimedean axiom*: For any two events E and E^*, define a sequence of events $A_1 \ldots A_j$, which are each disjoint from E and E^*. Suppose $m_1(A_1 \textit{ or } E)>m_1(A_1 \textit{ or } E^*)$. This could be interpreted as saying that E is more likely than E^*. Also suppose that $m_1(Aj_{+1} \textit{ or } E^*)=m_1\ (E \textit{ or } A_j)$ for each j. This could be interpreted as saying that each event in the sequence is less likely than the event following it. Consider the sequence of values $m_1(E^* \textit{ or } A_1)$, $m_1(E^* \text{ or } A_2)\ldots$. This sequence of numbers is only bounded if the number of events in the sequence $A_1 \ldots A_j \ldots$ is finite.
(6) *Ratio-scale*: m_1 is unique up to multiplication by an arbitrary constant. (This property is required since the probabilities are uniquely defined by basic probability assignments up to multiplication by an arbitrary constant.)

We now prove the following proposition:

Proposition. *Given the preceding six assumptions*

$$m_1(E) = \left[\sum_j [m_1(A_j)]^{1/k}\right]^k \quad \textit{for some constant } k.$$

Proof. Krantz et al. [34] show that given the axioms of additive conjoint measurement (which are implied by (1) through (5)), there exists a continuous function h such that

$$h(m_1(A)) = \left[\sum_j h(m_1(A_j))\right].$$

Given assumption 6, m_1 is unique up to an arbitrary positive scaling factor so that for any positive constant C

$$h(Cm_1(A)) = \sum_j h(Cm_1(A_j)).$$

Solving this functional equation gives the theorem.

We similarly attach the m_2 assignment to singleton events and then write the m_2 assignment for the disjunction of singletons as a function of the m_2 assignment of the singletons. Imposing the previous six assumptions on this function implies that

$$m_2(E) = \left[\sum_j [m_2(A_j)]^{1/k}\right]^k.$$

Since $m(E)$ is a simple average of $m_1(E)$ and $m_2(E)$, we get

$$m(E) = (1/2)\left\{\left[\sum_j [m_1(A_j)]^{1/k}\right]^k + \left[\sum_j [m_2(A_j)]^{1/k}\right]^k\right\}. \qquad \square$$

3.5. Application to quantum physics and cognitive psychology

We now replace our two secondary basic probability assignments by two support functions, s_1 and s_2, defined by

$$m_1(A_j) = [s_1(A_j)]^k \quad \text{and} \quad m_2(A_j) = [s_2(A_j)]^k.$$

Hence

$$m(E) = (1/2)\left\{\left[\sum_j s_1(A_j)\right]^k + \left[\sum_j s_2(A_j)\right]^k\right\}.$$

If all singletons have the same affinity for one another, $s = s_1 = s_2$ and we get

(1) $m(A) = s^k(A)$ for singleton events A,
(2) $s(E) = \sum_j s(A_j)$ if E consists of a union of the singletons $(A_1 \ldots A_n)$, and
(3) $m(E) = s^k(E)$.

This gives us (C1) and (C2), the rules for attaching basic probability assignments which leads to Kahneman and Tversky's subadditive support theory for $k < 1$. Note that $s(A)$ corresponds to what Tversky and Koehler called the support function.

Now suppose that the upper and lower basic probability assignments need not be equal. If $k = 2$, then we get

$$m(A) = (1/2)\{s_1^2(A) + s_2^2(A)\},$$

$$m(E) = (1/2)\left\{\left[\sum_j s_1(A_j)\right]^2 + \left[\sum_j s_2(A_j)\right]^2\right\}.$$

Note that $s_1(A)$ and $s_2(A)$ could be negative since our only constraint on support functions is that $m_1(A)$ and $m_2(A)$ be non-negative. We now define the complex wave function by

$$\psi(A) = (1/2)^{1/2}\{s_1(A) + (-1)^{1/2} s_2(A)\},$$

$$\psi(E) = (1/2)^{1/2} \sum_{A \text{ in } E} \{s_1(A) + (-1)^{1/2} s_2(A)\}.$$

Note that

$$m(A) = (s_1^2(A) + s_2^2(A))/2 = |\psi(A)|^2$$

and

$$m(E) = (1/2)\left\{\left[\sum_j s_1(A_j)\right]^2 + \left[\sum_j s_2(A_j)\right]^2\right\} = \left|\sum_j \psi(A_j)\right|^2,$$

which gives us (Q1) and (Q2), the rules for attaching basic probability assignments in quantum mechanics. Since the wave function reduces to what Tversky and Koehler called the support function when s_1 and s_2 are equal, we can interpret the wave function as a complex support function

Hence our restriction of Shafer's basic probability assignments leads to a formula that nearly coincides with empirical formulas in the unrelated fields of quantum physics and cognitive psychology. To show how this formula measures affinity, define

$$\theta(A) = \arctan(s_1(A)/s_2(A))$$

as a measure of the difference between the two support functions. This measure is zero in Tversky and Koehler's support theory. Since

$$m(A) = (1/2)\{s_1^2(A) + s_2^2(A)\},$$

we have

$$s_1(A) = (m(A))^{1/2}\cos(\theta(A)) \quad \text{and} \quad s_2(A) = (m(A))^{1/2}\sin(\theta(A)).$$

Now the previous quantum mechanical equation implied

$$m(E) = (1/2)\left\{\left[\sum_j (s_1^2(A_j) + s_2^2(A_j))\right] + 2\left[\sum_j \sum_{j*\,\text{not}\,j} [s_1(A_j)s_1(A_{j*}) + s_2(A_j)s_2(A_{j*})]\right]\right\}.$$

Substituting gives

$$m(E) = (1/2)\left\{\left[\sum_j m(A_j)\right] + 2\left[\sum_j \sum_{j*\,\text{not}\,j} [m(A_j)m(A_{j*})]^{1/2}\cos(\theta(A_j) - \theta(A_{j*}))\right]\right\}.$$

If $\theta(A_j)$ is close to $\theta(A_{j*})$, then $m(E)$ exceeds $[\sum_j m(A_j)]$ and the individual attaches a greater basic probability assignment to the disjunction than to the singletons. Conversely, when $\theta(A_j)$ is very different from $\theta(A_{j*})$, $m(E)$ is less than $[\sum_j m(A_j)]$ and the individual attaches a smaller basic probability assignment to the disjunction than either singleton. Hence $(\theta(A_j) - \theta(A_{j*}))$ is a measure of affinity between singletons A_j and A_{j*}.

4. An enhanced decision theory

4.1. Fuzzy satisficing

Expected utility theory indicates that individuals make choices maximizing a probability-weighted average of the utility of the possible consequences of those choices. It also presumes that any uncertain gambles have been defined so as to pass the clairvoyant test, i.e., all potential outcomes are observable. In this paper, we focused on utility assessment, probability assessment and the demands of the clairvoyant test.

Simon argued that individuals and firms do not optimize; instead they satisfice. Simon's challenge motivated the replacement of the utility function by the mathematically equivalent notion of a membership function in the fuzzy set *satisfactory*. Essentially, this replaces Savage's idea of optimizing utility by the idea of fuzzy '*satisficing*'. It means that instead of assessing utility functions, we should be assessing fuzzy membership functions.

4.2. Probabilities inferred from evidence

The Heisenberg Uncertainty Principle also requires a serious change in the conventional decision theoretic understanding of the clairvoyant test. This leads to a significant change in the decision theoretic understanding of probability (although, to some extent, it makes it more consistent with Kolmogorov's understanding of probability). Introducing the quantum logical notion of multiple incompatible experiments implies that there may be multiple inconsistent probabilities associated with different experiments.

As a result, the probabilities used in computing expected utility for gambles defined over different supports might differ. This can lead to a deviation in how expected utility theory ranks lotteries since expected utility theory presumes all probabilities are consistent. But, as we discussed previously, this presumption of consistency between all probabilities assumes that all the uncertain events associated with all possible decisions are simultaneously observable. And this is the assumption which the Heisenberg Uncertainty Principle denies. In such cases, our formulation differs from expected utility theory.

If these multiple sets of probabilities were totally unrelated, it might be possible to directly assess them independently. But there are some obvious interrelationships between these different fields. Hence probabilities cannot be assessed independently. We presented an assumption which implies that instead of assessing probabilities, we should assess 'the weight of evidence' for various states and then infer probabilities from those evidence weights. In this alternate procedure,

(1) Shaferian basic probability assignments are attached to all possible states. (These assignments are interpreted as the weight attached to various evidence.)
(2) The Borel fields associated with different experiments are specified.
(3) The states corresponded to the atoms of each Borel-field are identified. (This specifies which kinds of evidence are relevant to which kinds of experiments.)
(4) Probabilities are assigned to these atomic events using the basic probability assignments for the corresponding states. (Hence probabilities are derived from the weight attached to evidence.)

(5) Probabilities for non-atomic events in the Borel-field are computed as the sum of the probabilities for the atomic events composing them.

Six axioms were enunciated which specified how basic probability assignments were attached to the frame of discernment. This led to a model of uncertainty which coincides (up to a single unspecified parameter k) with empirically validated formulas in quantum physics and cognitive psychology.

To understand how this theory differs from other theories of non-additive probabilities, consider Schmeidler's theory [46,10] of non-additive probabilities. This theory retains Savage's focus on defining probabilities over the set of all subsets of S. But it restricts the application of Savage's sure-thing principle to co-monotonic acts. In contrast, we restrict the application of all of Savage's axioms to acts defined over the same field which is generally smaller than the power set of S. We then impose additional restrictions on our basic probability assignments. Thus, our formulation differs from these other formulations in focusing on the quantum logical possibility of multiple Borel-fields. A second difference is the close correspondence between our formulation and existing empirical formulas in quantum physics and cognitive psychology.

4.3. Practical implications

Hence our formulation proposes dramatic changes in how utilities and probabilities are assessed. This, in turn, leads to significant differences in the evaluation of decisions defined across different sets of observable events (i.e., gambles with different supports.) But our formulation has many other implications as well.

For example, consider how our probabilities will be updated in light of new information [27]. When we introduce new information, we frequently introduce new distinctions which may lead us to partition the events we are considering into smaller sub-events. De Finetti [11] described this as '*extending the conversation*'. In standard Bayesian theory, the subjective probabilities one assesses by '*extending the conversation*', i.e., by partitioning one's original events into sub-events, are consistent with the probabilities one had assessed prior to partitioning the event space.

But in our formulation, '*extending the conversation*' can cause our probabilities to change. (In the spirit of the Heisenberg Uncertainty Principle, the act of doing an experiment, i.e., of making new distinctions, has changed our probabilities.) The evidence compiled by Tversky and Koehler suggests that this will increase the probability assigned to the events in which greater distinctions are introduced.

In addition, of course, new information will provide data which will cause the probabilities assessed on this more refined space to be updated using Bayes' Rule. Hence, there are two impacts associated with collecting information:

(1) the updating of probabilities associated with using Bayes rule,
(2) the change in the Borel-field occasioned by 'extending the conversation'.

In many cases, the impact information has on the Borel-field used to understand our possibilities is much more important than the actual revision of probabilities associated with Bayes rule. In other words, a decision consultant can frequently best aid his client by helping the client understand the categories (or distinctions) which should be used in thinking about the world.

The economist Shackle [47, p. 75] had described uncertainty as the case in which "*the possible consequences of an act are not listable.*" Similarly in our case, there is no unambiguously correct Borel-field describing the possible outcomes of an action. Instead there are several possible Borel-fields from which the decision-maker implicitly chooses. In thinking about decision-making, Shackle [48, p. 96] wrote.

> *Choice does not consist in comparing the items in a list, known to be complete, of given fully specified rival and certainly attainable results. It consists in first creating by conjecture and reasoned imagination on the basis of mere suggestions offered…the things on which hope can be fixed.*

In a similar way, developing the Borel-field which one will use for analyzing the decision is a *creative* act that affects what decision is eventually found through analysis.

To illustrate the practical significance of these considerations, consider a corporation interested in developing new products for customers. The corporation could evaluate a product based on whether the market as a whole likes it or not. Or the corporation could focus on whether each one of the millions of customers in his market might like the product or not. In practice, the first way of thinking is too high level to be helpful; the second way of thinking is far too detailed to be helpful. As a result, the corporation typically groups individuals into various buying segments.

There are several possible ways of constructing segments, e.g., we could segment people into demographic groups (the young, the old and the middle-aged). In that case, our Borel-field would reflect the possible reactions of each of these groups to our product. Ford Motor Corporation tends to pursue this strategy. Or we could segment people in attitudinal groups (the conservatives, the liberals and the moderates). General Motors uses the latter approach. Hence, in this practical example, there is a choice about which Borel-field to use. This choice has profound implications for the kinds of product strategies which will be pursued by Ford and General Motors.

4.4. Summary

Quantum physics, cognitive psychology and Simon's organizational science have presented major challenges to the standard treatment of preferences and beliefs described in utility theory. In this paper, we present an extension of utility theory which responds to these challenges. This extension is:

(1) *Theoretically parsimonious*:

- We reinterpret the utility function instead of altering it.
- Without modifying Shafer's theory of evidence, we supplement it by introducing the notion of multiple Borel-fields and using his basic probability assignments to generate probabilities over those Borel-fields.
- ○ Without modifying the basic ideas of quantum logic, we supplement them by introducing Shafer's basic probability assignments defined over a frame of discernment underlying multiple Borel-fields.
- ○ We make the minimal modification in Savage required to accommodate the widely accepted anomalies of quantum physics.

(2) *A close fit with empirical data*:
- As Simon noted, there is widespread evidence that individuals, especially in organizations, seek to attain prespecified goals, and do not explicitly maximize value functions. Our formulation replaces utility-maximization with fuzzy goal-seeking.
- Evidence suggests that expected utility theory is only consistent with individual behavior when gambles all have the same support. Our formulation is consistent with expected utility theory when gambles all have the same support. It implies that expected utility is not applicable when gambles have different support, i.e., when utility theory no longer seems to fit empirical data
- Our deviation from expected utility theory is equivalent, up to a single scaling constant k, to empirically verified laws in quantum physics and cognitive psychology.

Hence, the resulting theory of fuzzy choice retains the normative appeal of expected utility theory while extending its empirical applicability.

Nonetheless, this paper has only presented the beginnings of a theory of fuzzy choice. There are a wide variety of questions in bounded rationality, quantum logic, quantum physics, cognitive psychology, and utility theory requiring further elaboration. This paper also introduced a new use of Shafer's theory of evidence which requires further investigation. In addition, our results needed to be extended to the case in which S is not atomic. We hope that this paper stimulates further work leading to the resolution of these and other issues.

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For future reading

The following articles could also be of interest to the reader, [3,4,6,9,14,22,23,24,26,29,30,31,37,58].

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