3B2v8.06a/w (Dec 5 2003).51c XML:ver:5.0.1
PHYSA : 8824
Prod.Type:FTP pp.1–18(col.fig.:NIL)
ED:Suraina PAGN:Dini SCAN:Nil
ARTICLE IN PRESS
Available online at www.sciencedirect.com
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Physica A ▮ (▮▮▮▮) ▮▮▮–▮▮▮
ELSEVIER
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www.elsevier.com/locate/physa

Econophysics and individual choice

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Received 30 March 2004; received in revised form 21 February 2005

Abstract

The subjectivist theory of probability specifies certain axioms of rationality which together lead to both a theory of probability and a theory of preference. The theory of probability is used throughout the sciences while the theory of preferences is used in economics. Results in quantum physics challenge the adequacy of the subjectivist theory of probability. As we show, answering this challenge requires modifying an Archimedean axiom in the subjectivist theory. But changing this axiom modifies the subjectivist theory of preference and therefore has implications for economics. As this paper notes, these implications are consistent with current empirical findings in psychology and economics. As we show, these results also have implications for pricing in securities markets.

This suggests further directions for research in econophysics.

PACS: 89.65.gh; 02.50.Cw; 03.65.−w

Keywords: Econophysics; Probability amplitude mechanics; Probability theory

1. Econophysics and psychology

As Roehner (2002, p. xiv) writes:

> There is currently a methodological gulf between econophysics and economics but there is also a deep cultural divide between economics and sociology which goes

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doi:10.1016/j.physa.2005.03.016

> back to the origins of the two disciplines ... by ignoring the borderlines between sociology and economics, econophysics has initiated a transformation.

But there is also a cultural divide between economics and psychology. This paper argues that econophysics, in addition to addressing the divide between sociology and economics (Yousefi & Rosser, 2003, [1,2]), can also address the divide between psychology and economics.

Mainstream microeconomics [3, p. 19] presumes that an individual makes 'rational' decisions on the basis of limited information and personal preferences. In the context of an uncertain world, the subjectivist Bayesian paradigm formalizes this notion of 'rationality' by requiring that rational decision-making satisfy seven consistency axioms. These axioms then lead to a mathematical theory describing the behavior of rational individuals.

While rational decision-making is of special interest in economics, the 2002 Nobel Prize in Economics to psychologist Daniel Kahneman recognized the involvement of psychology:

> Nearly half a century ago, Edwards introduced decision-making as a research topic for psychologists, outlining an agenda for future research. Allais outlined a psychology-based positive theory of choice under uncertainty, while Simon proposed an approach to information processing and decision-making based on bounded rationality. But research on cognitive psychology did not come into its own until Daniel Kahneman and Amos Tversky published their findings on judgement and decision-making. (Nobel Prize Citation, 2002.)

This psychological research identified systematic discrepancies between the economic theory of choice and what was observed in psychological experiments. Thus, the psychological evidence established that at least some of the subjectivist consistency axioms are violated in actual human behavior.

If these axioms are required for rational behavior, then all of the psychological evidence might reflect human irrationality. (And certainly much actual human decision-making is not fully rational.) However these same axioms imply that a rational individual will always (implicitly or explicitly) describe uncertainty using probability theory. Since this does not seem fully possible for quantum mechanical uncertainties, we conclude that at least one of these axioms cannot be a universal requirement for rational behavior. An extensive literature has carefully debated which axioms are universally required for rational behavior. Based on that literature, Fishburn and LaValle [4, p. 183] concluded that there is only one *axiom that can be violated in a prescriptively unassailable fashion*. This paper focuses on modifying this one non-normative axiom.

After the axiom is appropriately modified, the subjectivist paradigm implies probability amplitude mechanics in quantum physics. Hence the subjectivist Bayesian theory of probability now becomes consistent with probability theory as understood in quantum physics. But since this same non-normative axiom also underlies economics, this modification revises the subjectivist formulation of economics. As we will show, this revised theory also leads to some of the observed psychological behavior (e.g., see [5–8]) which was inconsistent with the subjectivist Bayesian paradigm.

Thus econophysics, in addition to addressing the split between economics and sociology, might also be capable of addressing the split between economics and psychology. The paper concludes with a discussion of its possible application to stockmarket behavior.

2. The subjectivist theory of probability

The conventional economic paradigm focuses on an individual choosing among alternative decisions f from some choice set D. To represent this individual's uncertainty about the consequences of different decisions, let s denote a possible state of the world with S being the set of all possible states. Then a decision can be represented as a function f with $f(s)$ denoting the consequence of decision f when the state of the world is s. The vector $\{f(s)|s \in S\}$ describes the consequences of the decision in all possible states of the world.

The individual's preferences over various decisions f (or, equivalently, various consequence vectors $\{f(s)|s \in S\}$) should be consistent with commonsensical notions of rationality. For example, suppose the individual preferred decision f to decision g and also preferred decision g to decision h. Then the *Transitivity Principle* implies that the individual should also prefer decision f to decision h. As another example, suppose that decisions f and g lead to exactly the same consequences when state s occurs. Then the *Sure-Thing Principle* implies that the payoffs associated with state s can be ignored in determining whether we prefer decision f to g. (Subjectivist arguments also show that when an individual's preferences violate these axioms of 'rationality', the individual will, in certain settings, make decisions that are clearly self-destructive.)

In his *Foundation of Statistics*, Savage [9] showed that these and other principles imply the existence of:

(1) A utility function $u(f)$ defined over each decision f with the individual always preferring that decision f with the larger value of $u(f)$. *This condition ensures that transitivity holds.*
(2) Part-worth functions $w(f, s)$ such that $u(f) = \sum_{s \in S} w(f, s)$. *This condition ensures that the Sure-Thing Principle holds.*
(3) A state-dependent function $p(s)$ and a consequence-dependent function $u^*(f(s))$ such that $w(f, s) = u^*(f(s))p(s)$. *This establishes the existence of a probability function*, $p(s)$.
(4) The equivalence of the consequence-dependent function $u^*(f(s))$ to the utility of a hypothetical act f^* giving the same consequence $c = f(s)$ in every state. *This establishes that probabilities sum to one*.

This subjective expected utility (SEU) theory integrates the theory of probability with the theory of rational choice.

As Savage notes, some examples of possible states s are *the infinite sequence of heads and tails that will result from repeated tossings of a fair coin*, *the complete decimal expansion of* π; *the exact entire past*, *present and future history of the universe understood in any sense however wide* [9, p. 8]. Hence Savage allows states to be intrinsically unobservable. Since his theory *does assume that probability is defined for all events*, *i.e.*, *for all sets of states* [9, p. 40], probabilities could be assigned to intrinsically unobservable events.

Savage recognized that *it is not usual to suppose*, *as has been done here*, *that* **all** *sets have a numerical probability* [9, p. 40]. For example, Kolmogorov's classical probability theory introduced a measurable set of subsets of S and only defined probabilities over that measurable set. (In this way, Kolmogorov avoided assigning probabilities to all subsets of the uncountable unit interval—which would have led to the Banach–Tarski Paradox [10]). Kolmogorov then interpreted the measurable set as the potential outcomes of an experiment and hence ensured that probabilities were only assigned to observable events.

Savage explicitly considered introducing measurable sets into his theory:

> *If one is unwilling to insist on comparisons between every pair of states or events*, *then in the same spirit*, *it is inappropriate to insist on comparisons between every pair of acts... All that has been*, *or is to be*, *formally deduced in this book concerning preferences among sets could be modified*, *mutatis mutandis*, *so that the class of events would not be the class of all subsets of S but rather ...a sigma-algebra on S*, *the set of all consequences would be a measurable space*, *that is*, *a set with a particular sigma-algebra singled out*, *and an act would be a measurable function from the measurable space of events to the measurable space of consequences.* [9, p. 42]

As he noted, implementing this change involves modifying his sixth axiom (the Archimedean Axiom). This axiom (with some paraphrasing) asserts:

The Archimedean Axiom. *Let s be a possible state of the world and consider any two actions f and g which give payoffs $f(s)$ and $g(s)$ in state s. Suppose the decision-maker prefers f to g. Now suppose we alter f by changing its payoff in state s from $f(s)$ to L (where*, *for example*, *L might be an extremely bad outcome*). *Let f^* be this altered act. If L is bad enough*, *intuition suggests the decision maker might now prefer g to f^*. But it is always possible to finitely partition S (thus reducing the likelihood of every state in S) so that the decision-maker still prefers f^* to g^*.* (**This rules out decision-makers who will refuse f^* as long as it has any non-zero chance of leading to the bad outcome.**) *And*, *conversely*, *suppose that instead of altering f*, *we alter g to g^* by replacing $g(s)$ by L where L is an extremely good outcome. A similar argument indicates that we can always partition S so finely that the decision-maker prefers f to g^*.*

Savage ultimately did not modify his Archimedean Axiom although he noted that *except for expository implications*, *it might have been mildly preferable to do so throughout* [9, p. 40]. The next section presents two reasons why this axiom should be modified.

3. Two objections to the Archimedean Axiom

3.1. The empirical objection

Edwards et al. [11] asked whether Savage's theory, intended for rational decision-makers, also approximately describes the actual behavior of intelligent decision-makers. We now consider one way in which psychologists found actual individual behavior deviating from Savage's model.

Suppose a subject is asked to assess the probability of death from natural causes. In thinking about this question, the subject might implicitly partition the set of possible occurrences into

$$b = (\text{death from natural causes}\,\text{death from unnatural causes})\,.$$

On the other hand, a subject asked to assess the probability of death from heart disease, cancer, and other natural causes might partition the set of possible occurrences into

$$b' = (\text{death from heart disease, death from cancer, death from other natural causes, death from unnatural causes})\,.$$

In Savage's theory, the probabilities assessed for death from unnatural causes should not be affected by whether the individual uses partition b or partition b'. But experimental evidence [12–17] establishes that assessed probabilities systematically vary between b and b'. In the previous example, the assessed probability for death from unnatural causes arising from partition b'—where the subject elaborated various causes of natural death—will usually be much less than the assessed probability arising from partition b.

This suggests that individuals who explicitly think about various causes of natural death (e.g. users of partition b') will consider it more likely than other individuals (e.g., users of partition b). Thus, the probabilities which individuals assess are partition-dependent. To mathematically describe the empirical evidence on partition-dependence, Tversky and Koehler [17] assigned support functions $q(s)$ to basic events s and then defined the support for a more aggregate event A by $q(A) = \sum_{s\in A} q(s)$. If b^* is a partition of events including A, then they define

$$p_{b^*}(A) = \frac{q^k(A)}{\sum_{A\in b^*} q^k(A)}$$

so that $p_{b^*}(A)$ can vary with b^*. Their model, and the empirical evidence, clearly violates Savage's theory.

But suppose we follow Savage in redefining his theory over a specific measurable set. Then the resulting probabilities and utilities are conditioned on that specific measurable set. As a result, changing the measurable set could potentially change the assessed probabilities. Since different partitions will induce different measurable sets, this allows probabilities to vary by partition. Bordley and Kadane [7] would refer to such probabilities $p_b(s)$ as 'experiment-dependent probabilities'.

3.2. The normative objection

The Archimedean Axiom does not allow for an individual who prefers f to f^* if either

(1) f outperforms f^* on some primary dimension,
(2) f and f^* yield the same payoffs on the primary dimension but f outperforms f^* on the secondary dimension.

But these kinds of *lexicographic* preferences are justifiable in many contexts:

- *Screening*: An individual, confronted with a large number of alternative choices, will often screen the alternatives on the basis of some fairly simple criterion and then choose among the remaining alternatives on the basis of more detailed criteria.
- *Constrained Optimization*: Consider the problem of optimizing some objective function subject to various constraints. This involves screening out solutions which violate the constraints and then evaluating the remaining solutions on the basis of their objective function value.
- *Ethical*: Many individuals, instead of simultaneously evaluating an alternative on the basis of its ethical and monetary implications, first reject those alternatives failing certain ethical criteria and then evaluate the remaining alternatives on the basis of monetary implications.

Thrall (1954) and Georgescu-Roegen [18] discussed other contexts where non-Archimedean preferences are appropriate.

Theories of rational choice recognize that there is nothing intrinsically irrational about these kinds of preferences. But given these preferences, no amount of improvement on the second dimension can compensate for a slight degradation on the first dimension. As a result, it is not possible to represent such preferences with a scalar-valued utility, $u(f)$. Hence theories of rational choice almost always rule out such preferences for mathematical convenience.

Hausner [19] and Chipman [20] showed that such preferences could be described by changing $u(f)$ to an n-element vector (for some n) and defining $u(f)>u(g)$ if, for any $k\geqslant 0$, the first k elements of $u(f)$ were identical to the first k elements of $u(g)$ and the $(k+1)$st element of $u(f)$ exceeded the $(k+1)$st element of $u(g)$. For scalar-valued probabilities, this utility function has the form

$$u(f) = \sum_s w(f,s) = \sum_s p(s)u(f(s)) ,$$

where $u(f)$ and $u(f(s))$ are now n-element vectors. But LaValle and Fishburn [21] noted that probabilities, in such contexts, can be matrix-valued. This paper specifically replaces $p(s)$ with the square $n\times n$ matrix $P(s)$.

4. Modifying the Archimedean Axiom

4.1. Integrating these two solutions

If the Archimedean Axiom is relaxed, then scalar probabilities $p(s)$ cannot always be directly defined over the state-space S. The preceding section discussed two alternative ways of defining probability with a relaxed Archimedean Axiom. One solution defines a sigma-field over S and replaces $p(s)$ by scalar partition-dependent probabilities $p_b(s)$ defined over the sigma-field instead of S. A second solution replaces Savage's scalar probabilities, $p(s)$, by matrix-valued probabilities, $P(s)$, defined over all states in S. This section integrates these two solutions.

We first prove:

Lemma 1. *There exists a non-negative function $m(s)$ such that the partition-dependent probability given partition b is*

$$p_b(s) = \frac{m(s)}{\sum_{s \in b} m(s)} \; .$$

Proof. Consider two decisions f and f^* which give the same payoff in states other than states s and s^*. Also consider two partitions, $b(1)$ and $b(2)$, which both include states s and s^*. Since s and s^* are elements of $b(1)$, the Sure-Thing Principle, applied to $b(1)$, indicates that the individual's preference between f and f^* will only depend on the payoffs of f and f^* in states s and s^*. Since s and s^* are elements of $b(2)$, the Sure-Thing Principle, applied to $b(2)$, likewise indicates that the individual's preference between f and f^* will only depend on the payoffs of f and f^* in states s and s^*. Hence if the individual prefers f to f^* given partition $b(1)$, the individual should prefer f to f^* given partition $b(2)$ (and vice versa). This implies [8] that

$$\frac{p_{b(1)}(s)}{p_{b(1)}(s^*)} = \frac{p_{b(2)}(s)}{p_{b(2)}(s^*)} \; .$$

In other words, the relative probabilities should be the same in both states. This leads to Lemma 1. (This lemma also leads to Tversky and Koehler's support theory if $m(s) = q^k(s)$.) $\square$

In order for the matrix-valued probabilities, $P(s)$—when restricted to a partition b—to be consistent with the scalar-valued probabilities, $p_b(s)$, we assume:

Condition 0. *For some matrix-valued function F, $m(s) = F(P(s))$.*

Whenever events A and B are independent with respect to the matrix-valued probability, it seems reasonable to require that they be independent with respect to the scalar-valued probability. As a result, we assume:

Condition 1. *$P(A\&B) = P(A)P(B)$ implies $m(A\&B) = m(A)m(B)$.*

Since probabilities are uniquely defined up to affine transformations and since each row of $P(A)$ corresponds to a probability, it also seems reasonable to assume:

Condition 2. *$m(A)$ is linear in the first row of $P(A)$.*

The function F would be trivial if changes in P never led to changes in m. Thus, we also assume:

Condition 3. *$F(P)$ sometimes changes if any two rows of P are changed.*

Together these conditions imply:

Lemma 2. *$F(P(s))$ is the determinant of $P(s)$.*

Proof. The first condition implies $F[P(A)P(B)] = F[P(A)]F[P(B)]$. Since condition 3 indicates that $F(P)$ cannot equal one for all choices of P:

- $F(PI) = F(P)$, where I is the identity, implies $F[I] = 1$.
- $F(0P) = F(0)$, where 0 is the zero matrix, implies $F[0] = 0$.
- $F(P^{-1}P) = F(I)$, for P an invertible matrix, implies $F(P^{-1}) = 1/F(P)$.

When P equals its own inverse, $F(P) = 1$ or $F(P) = -1$. For any m and n, consider a matrix I_{mn} which interchanges rows m and n of a matrix. Since it is its own inverse, $F(I_{mn}) = 1$ or -1. But if $F(I_{mn}) = 1$, then $F(P)$ never changes when rows m and n are interchanged, in conflict with Condition 3. Hence $F(I_{mn}) = -1$. Together with Condition 2 and $F(I) = 1$, these properties [22] imply that F is the determinant of the matrix.

4.2. *Probability amplitude mechanics*

In Savage's probability theory:

- The probability of the union of disjoint events is the sum of the probabilities.
- The joint probability of independent events is the product of the probabilities.
- Whenever A has non-zero probability, there exists a unique conditional probability for event E occurring given that event A has occurred. The joint probability of event E and A is the product of A's probability and this is conditional probability.

Scalar probabilities can have these properties because they are defined over the field of real numbers so that

- addition and multiplication are well-defined, commutative and associative, with a zero and an identity,
- an additive inverse exists for every quantity and a multiplicative inverse exists for every non-zero quantity.

We now similarly require that our matrix-valued probabilities be defined over a field. Since only fields involving finite-dimensional real-valued vectors are relevant for matrix-valued probabilities, results from Mishchenko and Solovyov [23] show that the required field must be isomorphic to either the real or complex numbers. The only set of matrices isomorphic to the real or complex numbers [24] has the form

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

for some real numbers α and β. Hence our matrix-valued probabilities must be defined over this family of matrices.

Since the determinants of these matrices are non-negative, Lemma 2 implies that $m(s)$ is non-negative. Given Lemma 1, these scalar probabilities must be real numbers ranging between zero and one. (Hence our formulation automatically ensures that scalar probabilities satisfy all of the standard properties of probability.)

Since many matrices have the same determinant, there are many different matrix-valued probabilities associated with the same measure $m(s)$. Define C as the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $C^2 = -I$. Then any matrix-valued probability, P, can be written as $P = \alpha I + \beta C$. When the matrix $P = \alpha I + \beta C$ is replaced with the corresponding complex number $\phi = \alpha + \beta \mathrm{i}$, the transpose of P corresponds to the complex conjugate of ϕ and the determinant of P to the complex square of ϕ. Thus $F[P(s)] = |\phi(s)|^2$ and

$$p_b(s) = \frac{|\phi(s)|^2}{\sum_{s \in b} |\phi(s)|^2} .$$

Since $P(s \vee s^*) = P(s) + P(s^*)$ when s and s^* are disjoint, we also have $\phi(s \vee s^*) = \phi(s) + \phi(s^*)$. Hence, as Youssef [25–27] and Srinivasan and Sudarshan [28] argued, the wave function ϕ is a complex probability.

Each state $s \in S$ is associated with a probability matrix $P(s)$ and a corresponding wave function $\phi(s)$. If an event E is the union of some of the states in partition b, then the probability of event E in that partition is

$$p_b(E) = \sum_{s \in E} p_b(s) .$$

If events s and s^* are not elements of the partition b but their union $s \vee s^*$ is, then

$$p_b(s \vee s^*) \propto |\phi(s \vee s^*)|^2 \propto |\phi(s) + \phi(s^*)|^2 .$$

Thus our formalism is isomorphic to probability amplitude mechanics [29].

5. Potential implications for individual preferences

5.1. Target-oriented preferences

Consider first the case of scalar utilities.

If an individual is solely concerned with achieving some target (e.g., winning the Nobel Prize), then utility can be scaled so that $u(c)$ equals one if c achieves the target and zero else. Hence the expected utility of a decision f which has various probabilities of leading to various consequences is just the probability of decision f leading to a consequence that achieves the target.

But in many cases, the individual does not know whether a specific consequence c will achieve the target:

- Many companies focus on outperforming the future performance of well-specified benchmark competitors [30] without knowing what that future performance will be.
- Many product designers focus on meeting future customer expectations while being uncertain of those future expectations [31].
- Many portfolio managers [32] are evaluated on how well their portfolio's financial return compares with the uncertain return of some benchmark stock index (like the S & P 500).
- Many individuals save money either for the college education of their children or for retirement and do not know how much will be required to meet this objective.

In these cases (Bordley & Kirkwood, 2004), the utility of consequence f is still the probability of the decision's consequences achieving a target.

In fact, Castagnoli and LiCalzi [33] and Bordley & Castiglione (2000) showed that Savage's utility function can always be interpreted as the probability of achieving a target. Specifically let $v(c)$ be the monetary value of consequence c and let X_f be a random variable which describes the uncertain monetary payoffs from decision f. Then there *always* exists a random variable T such that the expected utility of decision f is the probability of X_f exceeding T. This way of interpreting utility is especially useful when it is possible to empirically specify the stochastic threshold, T.

But it is also meaningful when T is poorly understood. For example, if the individual has little information about the target, then specifying the distribution of T to maximize entropy assigns T a uniform distribution. In this case the utility of decision f is

$$\begin{aligned}\int_s p(s)u(f(s)) &= \int_x \Pr(X_f = x)u(x) = \int_x \Pr(X_f = x)\Pr(x > T)\\ &= \int_x \Pr(X_f = x)x\end{aligned}$$

and maximizing expected utility is equivalent to maximizing expected monetary value—which is the most widely used objective function in the economic literature.

If the individual can specify the target's mean value, this same maximum entropy argument assigns T an exponential distribution and the utility of consequence c becomes exponential. In the decision analysis literature [34], the exponential utility function is widely regarded as the natural alternative to expected value when it is important to adjust for the individual's attitude toward risk.

Finally if the individual can specify both the mean and variance of the target, a maximum entropy argument assigns T a Gaussian distribution and the utility of consequence c becomes a cumulative normal distribution. This distribution is S-shaped about the mean of T and implies that the individual will prefer the sure thing when the sure thing exceeds the mean of T (i.e., exceeds his expected requirements) and prefer the gamble whenever the sure thing falls below the mean of T (i.e., falls below his expected requirements.) Friedman and Savage [35] and Markowitz [36] had long proposed similar S-shaped utility functions in order to explain why individuals simultaneously bought insurance and lottery tickets. Related empirical work by Kahneman & Tversky (1972) also argues that individuals act as if they maximized an S-shaped value function whose inflection point was called the 'reference point'.

In this formulation, the utility of reaching the target always equals one and the utility of any observable consequence c is always the probability of achieving that target. Hence utility, in the scalar case, can be treated as a probability. But the previous section replaced scalar probabilities with complex-valued probabilities. Hence both utility and probability become complex numbers.

5.2. The Allais Paradox

We now apply this approach to the 'Allais Paradox':

> Outside of the economics profession, Allais is perhaps best known for his studies of decision making under risk and what is generally known as the Allais Paradox. He used this paradox to show that the theory of maximization of expected utility ... which had been accepted by economists for more than 40 years, is contradicted by empirical observations of human behavior in some important decisions under risk. (1998 Nobel Prize in Economics Citation)

While Allais originally presented his paradox to an academic audience (including future Nobel Laureates Samuelson, Friedman, Arrow and Allais and distinguished statisticians like Savage and DeFinetti), variants of this same experiment have been replicated in hundreds of other contexts [37].

One of the more common variants [38] of Allais's experiment presents an individual with two sets of choices. The first set of choices asks individuals to state their preference between

(1) gamble 1, offering them a certain million dollars, and
(2) gamble 2, offering
  - an 89% chance of a million dollars,
  - a 10% chance of $2.5 million dollars and

- a 1% chance of nothing.

Many individuals, offered the choice between a million dollars for sure and a risky lottery, preferred the sure thing, gamble 1, to gamble 2.

These same individuals are then offered a choice between

(1) gamble 3, offering
  - an 89% chance of nothing and
  - an 11% chance of a million dollars, and

(2) gamble 4, offering
  - a 90% chance of nothing and
  - a 10% chance of \$2.5 million.

Many individuals, offered a choice between an 11% chance of a million dollars and a slightly lower chance of a much greater payoff, now preferred gamble 4 with the potentially higher payoff.

Unfortunately these seemingly plausible choices can be easily shown to violate Savage's utility theory. In fact, Savage himself, when presented with these choices, also violated his own utility theory by preferring gamble 1 to gamble 2 and gamble 4 to gamble 3. In response, Savage argued that individuals, like himself, who made these choices would have made more 'consistent' choices if the Allais gambles were presented in a different way. After defining events E_1, E_2, E_3 with probabilities 89%, 10%, 1 %, respectively, Savage proposed rewriting the Allais gambles as follows:

	E_1	E_2	E_3
Gamble 1	\$1 million	\$1 million	\$1 million
Gamble 2	\$1 million	\$2.5 million	\$0 million
Gamble 3	\$0 million	\$1 million	\$1 million
Gamble 4	\$0 million	\$2.5 million	\$0 million

The payoff probabilities from these choices are identical to the payoff probabilities in Allais's original gamble. But note that when event E_1 does not occur, gambles 1 and 3 both offer \$1 million for certain while gambles 2 and 4 offer \$2.5 million if event E_2 occurs (and nothing otherwise.) Hence if the individual prefers gamble 1 to gamble 2 when event E_1 does not occur, that individual should also prefer gamble 3 to gamble 4 when E_1 does not occur. But when event E_1 does occur, then gambles 1 and 2 offer the same payoff and gambles 3 and 4 offer the same payoff. Savage's Sure-Thing Principle indicates that the 'rational' individual ignores event E_1 in deciding between gambles 1 and 2 and in deciding between gambles 3 and 4. Hence if there is any chance of event E_1 not occurring, an individual who preferred gamble 1 to gamble 2 should also prefer gamble 3 to gamble 4. (And similarly an individual who preferred gamble 2 to gamble 1 should also prefer gamble 4 to gamble 3.)

Savage concluded that this representation of the gambles made the 'rational' choice more transparent and that 'rational' individuals (like himself), who violated utility theory in the original Allais experiment, would recognize their error and make different choices with his 'more transparent' representation.

But the adjusted theory proposed in this paper provides a slightly different explanation for Savage's conclusions. In the original Allais experiment, gambles were not explicitly defined over a common sigma-field while Savage's revised formulation explicitly defines a common sigma-field based on the events E_1, E_2, E_3. Since this paper only mandates utility theory (EU) when gambles are defined over a common sigma-field, we would only expect utility-maximizing (EU) behavior with Savage's revised gambles. In other words, our formulation allows 'rational' individuals to violate EU in the Allais gambles while requiring their choices to be consistent with EU when gambles are defined across a common sigma-field.

There is some empirical evidence related to this notion of rationality. Since Allais's work, many alternatives to Savage's model of choice have been proposed. Harless & Camerer (1998) reviewed thousands of empirical tests of Savage's expected utility theory (EU) versus other proposed models of how individuals choose. Since increasing the number of estimable parameters can increase a model's empirical fit without improving its predictiveness, they evaluated each model based on empirical fit and on the number of estimable parameters. Their article noted that

> there are dramatic differences between theory accuracy when the gambles in a pair have different supports and when they have the same support. EU predicts poorly when the support is different and predicts well when the support is the same ... When lotteries have different support, there is never a price-of-precision which justifies using EU.

Our formulation would typically allow violations of utility theory when lotteries have different support while demanding adherence to utility theory when lotteries have the same support.

As previously noted, there are many other proposed explanations of the Allais Paradox. For example, an individual who chooses gamble 2 and ends up with nothing may feel very badly (or be criticized by his peers) for deciding to forsake a guaranteed chance of a million dollars. To avoid the possibility of feeling foolish, he might choose gamble 1. In contrast, an individual who chooses gamble 4 and ends up with nothing can reassure himself with the fact that gamble 3 also had a very high chance of giving him nothing. Hence he chooses gamble 4. Kahneman & Tversky (1974)'s seminal paper also postulated a *certainty effect* which inclines individuals to prefer a certain outcome over risky outcomes far more than utility theory would suggest. More generally, individuals frequently act as if there is a big difference between no chance of zero and a 1% chance of zero but little difference between an 89% chance of zero and a 90% chance of zero.

These explanations involve violating one of Savage's prescriptively defensible axioms (like the Sure-Thing Principle). The main point of this section is that even individuals satisfying Savage's prescriptively defensible axioms (without the

Archimedean Axiom) can violate utility theory in the Allais experiment while satisfying utility theory when gambles are defined across a common sigma-field.

6. Implications for econophysics

6.1. Implications for rational behavior

Hence making a rational choice between gambles requires a (possibly implicit) specification of a common sigma-field underlying those gambles. Is there some *right* approach toward specifying the sigma-field in analyzing a complex choice?

One approach, which follows Savage's original formulation, is to define a sigma-field detailed enough to describe any possible occurrence. We then use the probabilities defined over such a *grand* sigma-field to infer the appropriate probabilities for the coarser *small-world* sigma-field relevant in addressing a particular decision problem. But as Foulis and Randall [39] wrote, no such *grand* sigma-field exists:

> *The grand canonical measurement of classical mechanics permits one to determine simultaneously the location and the momentum of all the particles of a physical system ... in quantum mechanics, the celebrated Heisenberg commutation rules reject both determinism and even an in principle possibility of a grand canonical measurement. Thus in quantum mechanics, we are denied the convenience of a single classical sample space in terms of which we are always able to confirm or refute the measurement of an observable.*

Another, more practical, approach is to directly specify the *small-world* sigma-field as the coarsest sigma-field detailed enough to describe the gambles with which the decision maker expects to be confronted in a given situation. Given this approach, individuals might make the choice between Allais's gambles 1 and 2 by defining a sigma-field based on three events and make the choice between Allais's gambles 3 and 4 by defining a sigma-field based on two events. But as we now show, this plausible heuristic means that what individuals choose can be affected by the order in which they are confronted with choices.

Consider the following two-part experiment: The subject is first presented with gambles defined over a detailed partition and evaluates these gambles by specifying a detailed sigma-field consistent with this detailed partition. The subject is then presented with gambles defined over a more aggregate version of the original partition and continues to use this detailed sigma-field in evaluating the gambles. Since all gambles are defined over the same sigma-field, a subject's choices across the first and second part of the experiment will be consistent with expected utility.

Now consider the *reversed* two-part experiment: The subject is first presented with gambles defined over an aggregate partition and evaluates them by specifying a more aggregate sigma-field. The subject is then presented with gambles defined over the detailed partition. Since the detailed gambles cannot be defined over the aggregate sigma-field, the subject replaces this sigma-field with a more detailed sigma-field and makes choices accordingly. Since the gambles in the first and second parts of the experiment are defined over different sigma-fields, the subject's choices across both parts of the experiment could violate expected utility. (Of course, if the subject is presented, once again, with the gambles defined over the aggregate partition, the subject might now evaluate them using the detailed sigma-field—which could reverse his preferences for the original aggregate gamble.)

Hence reversing the order in which choices are presented could lead to different choices. We now discuss some possible implications for stockmarket analysis.

6.2. Implications for investment behavior

Investment theory [40] defines an investment security as a random variable defined over a traditional probability space (S, F, P) where F is a sigma-field of subsets of S and P is defined over F. Thus, the payoffs of each security are only defined over events in F and not over states in S. In the context of project management, Pich et al. [41, p. 1011] referred to F as the *set of all events whose occurrence or non-occurrence can be anticipated, and hence planned for, by the project team* with events not definable over F sometimes corresponding to what is informally called *unknown unknowns or unk-unks*.

We similarly interpret F as the set of all events which can be anticipated in the stockmarket. Following Marshak and Radner [42], we call this sigma-field F *payoff-adequate* with respect to the set of possible investor actions and preferences if no investor anticipates any benefit from further decomposition of the field. Conventional *no arbitrage* arguments imply that security prices—defined over this sigma-field—must be consistent [43,44].

As previously noted, Savage's definition of S includes every possible occurrence (e.g., *the exact past, present and future history of the universe*) and hence includes events that are not anticipated by a stockmarket with a finite number of stock securities. Thus the market sigma-field F will only include a small portion of all the events definable in S.

As a result, changing economic circumstances could lead to the introduction of new securities whose payoffs depend on events not previously anticipated in the stockmarket and which cannot be mechanically priced in terms of existing stockmarket securities. (Or, alternatively, changing economic circumstances could make certain securities—and the associated events over which they were defined—irrelevant.) Such changes undermine the adequacy of the original market sigma-field. Inadequate sigma-fields are common in other problem areas, e.g., *in complex projects, an adequate representation of all the states that might have a significant influence on the project payoff, or of the causal relationships, may simply be beyond the capabilities of the project team* [41]. Schrader et al. [45] refer to this kind of sigma-field inadequacy as *ambiguity*.

Over time, a new sigma-field emerges which is more appropriate for the new set of economically relevant securities. Given this new sigma-field, a new set of equilibrium prices arises. In the absence of the Archimedean Assumption, the probabilities assigned to events in this new sigma-field need not be consistent with the probabilities in the previous sigma-field (just as the second set of probabilities in the reversed two-part experiment need not be consistent with the first set of probabilities). This paper drew on quantum mechanical principles to suggest how probabilities will change. Since stock prices are defined with respect to risk-neutral probabilities (Baxter & Rennie, 1996) and since risk-neutral probabilities must also be defined over the new sigma-field, stock prices will likewise change.

In our view, econophysics is a natural formalism for describing such changes.

7. Uncited references

[46–64].

References

[1] R. Mantegna, E. Stanley, An Introduction to Econophysics, Cambridge University Press, Cambridge, 1999.
[2] K. Ilinski, Physics of Finance: Gauge Modeling in Non-Equilibrium Pricing, Wiley, New York, 2001.
[3] M. Parkin, Economics, Addison-Wesley, Reading, MA, 1990.
[4] Fishburn, P. LaValle, Subjective expected lexicographic utility: axioms and assessment, Ann. Oper. Res. 80 (1998) 183–206.
[5] R. Bordley, Experiment-dependent probabilities in quantum mechanics & psychology, Phys. Essays 10 (1997) 3.
[6] R. Bordley, Quantum mechanical & human violations of compound probability principles: toward a generalized Heisenberg uncertainty principle, Oper. Res. (1998).
[7] R. Bordley, J. Kadane, Experiment-dependent priors in psychology & physics, Theory Decision (1999).
[8] R. Bordley, Reformulating decision theory using fuzzy set theory and Shafer's theory of evidence, Fuzzy Set Theory Syst. (2003).
[9] J. Savage, The Foundations of Statistics, Wiley, New York, 1954.
[10] S. Wagon, The Banach–Tarski Paradox, Cambridge University Press, Cambridge, 1985.
[11] W. Edwards, H. Lindman, L. Savage, Bayesian statistical inference for psychological research, Psychol. Rev. 70 (1963) 193–242.
[12] B. Fischoff, P. Slovic, S. Lichtenstein, Fault-trees: sensitivity of estimated failure probabilities to problem representation, J. Exp. Psychol.: Human Percep. Perform. 4 (1978) 330–344.
[13] D. Redelmeier, D. Kohler, V. Liberman, A. Tversky, Probability judgements in medicine: discounting unspecified alternatives, Med. Decis. Making 15 (1995) 226–230.
[14] E. Johnson, J. Hershey, J. Meszaros, H. Kunreuther, Framing, probability distortions & insurance decisions, J. Risk Uncertainty 7 (1993) 35–51.
[15] C. Fox, A. Tversky, A belief-based account of decision-making under uncertainty, in: D. Kahneman, A. Tversky (Eds.), Choices, Values & Frames, Cambridge University Press, Cambridge, 1995.
[16] Y. Rottenstreich, A. Tversky, Unpacking, repacking & anchoring: advances in support theory, Psychol. Rev. 2 (1997) 406–415.
[17] A. Tversky, D. Koehler, Support theory: a nonextensional representation of subjective probability, Psychol. Rev. 101 (1994) 547–567.
[18] N. Georgescu-Roegen, Choice, expectations & measurability, Quart. J. Econ. 68 (1954) 503–534.
[19] M. Hausner, Multidimensional utilities, in: R. Thrall, C. Coombs, R. Davis (Eds.), Decision Processes, Wiley, New York, 1954, pp. 167–180.
[20] J. Chipman, The foundations of utility, Econometrica 28 (1960) 193–224.

[21] LaValle, I.P. Fishburn, Subjective expected lexicographic utility: axioms & assessment, Ann. Oper. Res. 80 (1998) 183–206.
[22] H. Schneider, Matrices & Linear Algebra, Thomson Publishing, 1968.
[23] A. Mishchenko, Y. Solovyov, Quaternions, Quantum 11 (2000) 4–7, 18.
[24] L. Ahlfors, Complex Analysis, McGraw-Hill, New York, 1966.
[25] S. Youssef, Quantum mechanics as an exotic probability theory, in: K.M. Hanson, R.N. Silver (Eds.), Proceedings of the 15th International Workshop on Maximum Entropy and Bayesian Methods, Santa Fe, August 1995.
[26] S. Youssef, Is quantum mechanics an exotic probability theory?, in: D.M. Greenberger, A. Zeilinger (Eds.), Fundamental Problems in Quantum Theory; Conference in Honor of Professor John A. Wheeler, Ann. New York Acad. Sci., 755 (1995).
[27] S. Youssef, Is complex probability theory consistent with Bell's theorem?, Phys. Lett. A 204 (1995) 181.
[28] S.K. Srinivasan, E.C.G. Sudarshan, Complex measures and amplitudes, generalized stochastic processes & their application to quantum mechanics, J. Phys. A Math Gen. 27 (1994) 39.
[29] R. Feynman, Lectures in Physics, Addison-Wesley, Reading, MA, 1970.
[30] R. Bordley, Relating value-focused thinking and interactive planning, J. Oper. Res. Soc. (2001).
[31] R. Bordley, Integrating gap analysis and utility theory in service research, J. Service Res. (2001).
[32] A. Admati, P. Pfleiderer, Does it all add up? Benchmarking the compensation of active portfolio managers, J. Business 70 (3) (1997) 1323–1350.
[33] E. Castagnoli, M. LiCalzi, Expected utility without utility, Theory Decision 41 (1996) 281–301.
[34] R. Howard, Decision analysis: practice & promise, Manag. Sci. 34 (1988) 679–695.
[35] Friedman, L. Savage, The utility analysis of choices involving risk, J. Polit. Econ. 56 (1948) 279–304.
[36] H. Markowitz, The utility of wealth, J. Polit. Econ. 60 (1952) 151–158.
[37] M. Allais, O. Hagen, Expected Utility Hypothesis and the Allais Paradox, Reidel, Dordrecht, 1979.
[38] H. Raiffa, Introductory Lectures on Choices under Uncertainty, Addison-Wesley, Reading, MA, 1968.
[39] D. Foulis, C. Randall, Operational statistics, I: basic concepts, J. Math. Phys. 13 (1972) 11.
[40] R. Elliot, E. Kopp, Mathematics of Financial Markets, Springer, New York, 1991.
[41] M. Pich, C. Lock, A. DeMeyer, On uncertainty, ambiguity and complexity in project management, Manag. Sci. 48 (8) (2002) 1008–1023.
[42] J. Marshak, R. Radner, The Economic Theory of Teams, Yale University Press, CT, 1972.
[43] K. Arrow, The Role of Securities in the Optimal Application of Risk-Bearing, Rev. Econ. Stud. 31 (1964) 91–96.
[44] G. Debreu, The Theory of Value, Wiley, New York, 1959 (Chapter 7).
[45] S. Schrader, W. Riggs, R. Smith, Choice over uncertainty and ambiguity in technical problem-solving, J. Eng. Technol. Manag. 10 (1993) 73–99.
[46] P. Billingsley, Probability and Measure, Wiley, New York, 1995.
[47] M. Birnbaum, T. Martin, Generalizing across people, procedures and predictions: violations of stochastic dominance and coalescing, in: Schneider, J. Shanteau (Eds.), Emerging Perspectives on Decision Research, Cambridge University Press, New York, 2003.
[48] R. Bordley, M. LiCalzi, Decision analysis using targets instead of utility functions, Decisions Econ. Finance 23 (2000) 53–74.
[49] R. Bordley, Relating probability amplitude mechanics to standard statistical models, Phys. Lett. A 204 (1995) 26–32.
[50] A. Einstein, B. Podolsky, N. Rosen, Can the quantum mechanical description of physical reality be considered complete?, Phys. Rev. 47 (1935).
[51] C. Fox, Y. Rottenstreich, Partition priming in judgement under uncertainty, Psychol. Sci. 14 (2003) 3.
[52] C. Fox, K. See, in: D. Hardman, L. Macchi (Eds.), Thinking: Psychological Perspectives on Reason, Judgement and Decision Making, Wiley, New York, 2003.
[53] D. Harless, C. Camerer, The predictive utility of generalized expected utility theories, Econometrica 62 (1994) 1251–1290.

[54] J. Hirshleifer, Investment decision under uncertainty: applications of the state-preference approach, Quart. J. Econ. 80 (1966) 252–277.
[55] D. Kahneman, A. Tversky, Prospect theory: an analysis of decision under risk, Econometrica (1979).
[56] D. Kahneman, D. Slovic, P.A. Tversky, Judgement under uncertainty: heuristics and biases, Science 185 (1974) 1124–1131.
[57] D. Kahneman, P. Slovic, A. Tversky, Judgement under Uncertainty: Heuristics and Biases, Cambridge University Press, Cambridge, 1982.
[58] A. Kolmogorov, Foundations of the Theory of Probability, Chelsea, Bronx, New York, 1956.
[59] G. Loomes, R. Sugden, Regret theory: an alternative theory of rational choice under uncertainty, Econ. J. 92 (1982) 805–824.
[60] M. Schervish, T. Seidenfeld, J. Kadane, State-dependent utility, J. Am. Stat. Assoc. 85 (1990) 840–847.
[61] S.K. Srinivasan, Quantum mechanics via extended measures, J. Phys. A 23 (1997) 8297.
[62] S.K. Srinivasan, Complex measure, coherent state and squeezed state representation, J. Phys. A (2002).
[63] S. Youssef, Quantum mechanics as complex probability theory, Mod. Phys. Lett. A 9 (1994) 2571.
[64] S. Youssef, A reformulation of quantum mechanics, Mod. Phys. Lett. A 6 (1991) 225–236.