Consolidating Distribution Centers Can Reduce Lost Sales

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Abstract

This paper focusses on industries in which product is sold out of inventory by local retailers. As we show, when retailers are uncertain about the product's demand, expected product sales will fall short of expected product demand by an amount proportional to the standard deviation of demand. This implies that consolidating retailer outlets — in a way that does not reduce overall demand — reduces expected lost sales.

1 INTRODUCTION

In many industries, product is distributed to retailers who then sell out of finite product inventories. Because replenishing inventories is frequently time-consuming and because customers are frequently impatient, inventory shortages often result in lost sales. As this paper shows, the expected lost sales — assuming a profit-maximizing retailer — is proportional to the standard deviation of the retailer's demand uncertainty[1]. This implies that any consolidation of retailer outlets — which does not reduce product demand — will reduce the expected number of inventory shortages and thus the expected number of lost sales.

To derive this result, our first section generalizes the standard safety stock model of inventory theory to the multiple-retailer case. In addition to deriving a multiple-retailer generalization of the safety stock formula, we also deduce expected lost sales as proportional to the standard deviation of the retailer's demand uncertainty. We then use this relationship to derive a relationship between expected lost sales and the number of retailer outlets.

[1] This, incidentally, provides a way to quantify the cost of demand uncertainty. In many cases, wholesalers are unwilling to expend significant amounts of money on reducing their retailer's demand uncertainty because they fail to realize how much it impacts their own sales.

2 The Generalized Safety Stock Model

2.1 The Multiple-Retailer Safety Stock Model

The single period newsvendor model is the fundamental building block of many other more realistic inventory models(see, e.g., Porteus(1990)). In the newsvendor model, a retailer observes his stock on hand at the start of some period, places an order and receives a delivery. We let I_i denote the total inventory (stock on hand plus deliveries) of product which dealer i has available at the start of some typical period. Over the course of the period, random demands, D_i, arrive and partially (or completely) deplete this inventory.

Now in the classic newsvendor model, individuals intending to buy product from dealer i who find product out of stock simply drop out of the market. We generalize the model by assuming that if a retailer i intender finds the retailer out of stock, he

- switches to dealer j with probability F_{ji}. If he finds dealer j out of stock, then — since he has already incurred the transportation costs associated with driving to dealer j — he backorders product from dealer j.
- backorders from dealer i with probability F_{ii}
- drops out of the market with probability F_{0i}

Then if dealer i's inventory at the start of the period is I_i, his sales, S_i, are given by the minimum of the amount of inventory of product available and the demand for product i supplemented by demand diverted from other retailers, i.e.,

$$\begin{aligned} S_i &= \max(D_i, I_i) + \sum_j F_{ij} \max(D_j - I_j, 0) \\ &= D_i + \sum_j (F_{ij} - \delta_{ij}) \max(D_j - I_j, 0) \end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and equals zero otherwise. Taking expectations gives

$$E[S_i] = E[D_i] + \sum_j (F_{ij} - \delta_{ij}) E[\max(D_j - I_j, 0)] \qquad (1)$$

In a straightforward extension of the classical safety stock model(Eppen & Martin,1988), we assume

ASSUMPTION 1: The demand for product at different retailers only varies by centering and scaling parameters so that

$$D_i = E[D_i] + \sigma_i \delta_i$$

where $\delta_i, i = 1, ..., n$ are identically distributed random variables with mean zero, variance one and constant correlation with one another.

Furthermore, $\Pr(\delta_i \geq x)$ **is strictly decreasing in** x**.**
This assumption includes the case in which demand distributions only differ by mean-preserving spreads (see, e.g., Gerchak & Mossman,1992).

If we define $m_j = E[\max(\frac{D_j - I_j}{\sigma_j}, 0)]$ and $L_i = E[D_i] - E[S_i]$ to be lost sales, then

$$L_i = \sum_j \sigma_j(\delta_{ij} - F_{ij})m_j$$

Now in the standard safety-stock model, the retailer sets his starting inventory levels so that

$$I_j = ED_j + K\sigma_j \tag{2}$$

which implies

$$L_i = m\sum_j \sigma_j(\delta_{ij} - F_{ij}) = m[\sigma_i - \sum_j F_{ij}\sigma_j] \tag{3}$$

In other words, retailer i's lost sales are proportional to his demand's standard deviation less the expected standard deviation of demand for retailers for whom his outlet is a second choice.

Before discussing the implications of this result, the next section need to verify that equation(2) — the safety-stock result — holds in the multiple-retailer case. conditions under which (2) holds in the multiple retailer case, thus establishing equation(3).

3 The Safety Stock Model in the Multiproduct Case

The newsvendor(Porteus,1990) is assumed to myopically optimize an objective function which is linear in the expected sales at a given inventory level, the starting inventory, mean demand less expected sales (i.e., the demand unmet from inventory), the amount of inventory sold, and expected sales less inventory (i.e., the amount of unsold inventory at the end of the period.) Given the appropriate specification of objective function, this myopic policy(Veinott,1965) will, under certain conditions, maximize the present value of the vendor's profit over an infinite time period.

Let P_i be the price retailer i charges. If we neglect terms not involving P_i or I_i, then these assumptions have the vendor optimizing

$$\Pi_i = [(P_i - C_i)E[S_i] - h_i I_i]$$

where C_i reflects the costs of selling inventory, the salvage value of unsold inventory and the possible loss in goodwill from unmet demand and h_i reflects the cost of acquiring inventory less the salvage value of inventory(holding costs.)

The vendor adjusts prices and inventories to satisfy the following first-order conditions:

$$\frac{\partial \Pi_i}{\partial I_i} = 0 \Longrightarrow (P_i - C_i)\frac{\partial E[S_i]}{\partial I_i} = h_i \tag{4}$$

$$\frac{\partial \Pi}{\partial P_i} = 0 \Longrightarrow \frac{\partial E[S_i]}{\partial P_i} = -\frac{E[S_i]}{P_i - C_i} \tag{5}$$

3.1 Inventory-Setting Behavior

We focus first on equation(5). Differentiating equation(1) by I_k gives

$$\frac{\partial E[S_i]}{\partial I_k} = (F_{ik} - \delta_{ik})\frac{\partial E[\max(D_k - I_k, 0)]}{\partial I_k} = (\delta_{ik} - F_{ik})\Pr(D_k \geq I_k) \tag{6}$$

Substituting into (5) gives

$$(P_i - C_i)(1 - F_{ii})\Pr(D_i \geq I_i) = h_i \Longrightarrow$$

$$\frac{h_i}{(P_i - C_i)(1 - F_{ii})} = \Pr(D_i \geq I_i)$$

We can think of $\pi_i = (P_i - C_i)(1 - F_{ii})$ as the marginal profit associated with selling a unit of product i — adjusted for backordering. Following standard assumptions in the literature, we assume that $\frac{h_i}{\pi_i}$ is a constant (since both are markups from the cost of the commodity being sold). This implies that $\Pr(D_i \geq I_i)$ is a constant across all retailers.

3.2 Constancy of Stockout Probability

Thus each vendor, regardless of the demands they are facing, make their pricing and inventory decisions so the the expected probability of stocking out is the same. If they were all successful, of course, the actual "number of days of inventory" would be the same for all retailers. This is not true, empirically, because vendors(or manufacturers) underestimate the demand for various products and fail to adjust prices and supplies quickly enough to equalize number of days of inventory. What **is** true empirically is that a retailer with a 'large number of days of inventory', relative to his peers, generally feels pressure to reduce that inventory by cutting prices and postponing further orders while a small "number of days of inventory" motivates the opposite behavior. This indicates that wholesalers and retailers do undertake to curb disparities in 'number of days of inventory', suggesting that they do try to approximately equalize the probability of stockout across retailers.

The constancy of stockout result implies that

$$\Pr(D_i \geq I_i) = \Pr(\frac{D_i - ED_i}{\sigma_i} \geq \frac{I_i - ED_i}{\sigma_i}) = \Pr(\delta D_i \geq \frac{I_i - ED_i}{\sigma_i})$$

is a constant for all i. Given the strict monotonicity of $\Pr(D_i \geq x)$, we conclude that $\frac{I_i - ED_I}{\sigma_i}$ is a constant, i.e.,

$$I_i = ED_i + K\sigma_i \tag{7}$$

which is the well-known safety stock formula. As noted before, this gives (5)

4 A Lemma for Consolidating Distribution Channels

Rewriting(5) gives

$$L_i = m(\sigma_i - \sum_j F_{ij}\sigma_j)$$

If we let G be the set of all retailers owned by a given wholesaler, then the total expected lost sales to the wholesaler is just

$$\begin{aligned} L_I &= \sum_{i\in G} E[D_i] - \sum_{i\in G} E[S_i] = m[\sum_{i\in G}\sigma_i - \sum_{i\in G,j} F_{ij}\sigma_j] \\ &= m[\sum_{i\in G}\sigma_i - \sum_j F_{Gj}\sigma_j] = m[\sum_{i\in G}\sigma_i(1-F_{Gi}) - \sum_{j\notin G} F_{Gj}\sigma_j] \end{aligned}$$

Now suppose the item could be sold through a single large vendor. If the large vendor could still attract the same demand as all the small vendors, then his expected demand would be $E[\sum_{i\in G} D_i]$ while the variance of his demand would be $Var[\sum_{i\in G} D_i] = \sum_{ij\in G}\rho_{ij}\sigma_i\sigma_j$ with ρ_{ij} representing the correlation between product i and product j's demand. Then the expected lost sales from the large vendor would be

$$L_G = m[Var^{1/2}[\sum_{i\in G} D_i] - \sum_{j\notin G} F_{Gj}\sigma_j] = L_I + m[[\sum_{ij\in G}\rho_{ij}\sigma_i\sigma_j]^{1/2} - \sum_{i\in G}\sigma_i(1-F_{Gi})]$$

If we define $\rho = \frac{\sum_{i\neq j}\sigma_i\sigma_j\rho_{ij}}{\sum_{i\neq j}\sigma_i\sigma_j}$ to be the average correlation of demand between different retailers, $c = \frac{\sum_i \sigma_i^2}{(\sum_i \sigma_i)^2}$ to be an average coefficient of variation and $1 - F = \frac{\sum_i \sigma_i(1-F_{Gi})}{\sum_i \sigma_i}$ to be the average second choice substitutability among products, then

$$L_G = L_I + m[\sum_i \sigma_i^2(1-\rho) + \rho(\sum_i \sigma_i)^2]^{1/2} - (1-F)\sum_i \sigma_i] = L_I + m(\sum_i \sigma_i)[\rho + (1-\rho)c]^{1/2} - (1-F)]$$

If all retailers confront identical demands, $\rho = 1$ and $L_G - L_I \propto F$. is no benefit from consolidation. As ρ decreases, the reduction in lost sales becomes substantial. In fact, in the case in which retailer demand is identically distributed, our formula becomes

$$L_{large} = L_I(\frac{1}{n} + (1 - \frac{1}{n})\rho)^{1/2}$$

so that lost sales decrease with the inverse square root of the number of retailers being consolidated.

Thus consolidating distribution areas — if it can be done without decreasing product demand — will potentially increase overall sales.

Furthermore the corollary also seems to approximately hold when Assumption 1 is violated and the probability of stocking out is small (see Bordley, Beltramo & Blumenfeld,1993).

5 Conclusions

This paper showed that consolidating distribution channels will, by reducing expected inventory shortages, reduce lost sales. Hence reducing the standard deviation of demand will increase sales. Two ways of reducing the standard deviation of demand include:

- providing the newsvendor with improved forecasting techniques.
- eliminating any unreliability in the delivery shipments to the vendor. To understand why this reduces demand uncertainty, note that unreliable delivery shipments make I_i a random variable. Since the vendor's lost sales are determined by $D_i - I_i$, adding uncertainty to I_i is equivalent to adding uncertainty to the vendor's demand distribution (and increasing σ_i.) Hence reducing unreliability in deliveries reduces the 'effective' uncertainty in the vendor's demand distribution.

In the case of General Motors, this result justified the construction of an expensive software program for reducing the uncertainty in retailer inventories. It also provided further encouragement for providing sophisticated forecasting packages to GM's dealers.

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APPENDIX

Consider the class of models:

$$E[S_i] = \frac{a_i + \sum_{k=1}^{n} a_{ik} g_k(P_k) + \sum_{j,k=1}^{n} a_{ijk} g_j(P_j) g_k(P_k)}{b_0 + \sum_{k=1}^{n} b_k g_k(P_k) + \sum_{j,k=1}^{n} b_{jk} g_j(P_j) g_k(P_k)} \tag{8}$$

where g_j is an arbitrary monotonically increasing function of product j's price, $a_i, a_{ik}, a_{ijk}, b_0, b_k, b_{jk}$ are constants and $a_{ijk} = b_{jk} = 0$ for $j = k$.

First note that

$$\frac{\partial E[S_i]}{\partial P_j} = g'(P_j) \frac{a_{ij} - E[S_i]b_j + \sum_k g_k(a_{ijk} - q_i b_{jk})}{b_0 + \sum_k b_k g_k + \sum_{jk} b_{jk} g_j g_k}$$

Thus

$$\frac{\partial E[S_i]/\partial P_j}{\partial E[S_j]/\partial P_j} = \frac{a_{ij} - E[S_i]b_j + \sum_k g_k(a_{ijk} - E[S_i]b_{jk})}{a_{jj} - E[S_j]b_j + \sum_k g_k(a_{jjk} - E[S_i]b_{jk})} \tag{9}$$

Now let $n_i = a_i + \sum_k a_{ik} g_k + \sum_{jk} a_{ijk} g_j g_k$ and $d = b_0 + \sum_k b_k g_k + \sum_{jk} b_{jk} g_j g_k$ so that $E[S_i] = \frac{n_i}{d}$. Then the demand for product i if j became unavailable is

$$\frac{n_i - a_{ij} g_j - \sum_k a_{ijk} g_k g_j}{d - b_j g_j - \sum_k b_{jk} g_k g_j}$$

Subtracting off $E[S_i] = \frac{n_i}{d}$ gives the change in demand for i when j becomes unavailable. This gives

$$g_j \frac{a_{ij} - E[S_i]b_j + \sum_k g_k(a_{ijk} - E[S_i]b_{jk})}{d - g_j(b_j + \sum_k b_{jk} g_k)}$$

Dividing this by the change in j demand when j becomes unavailable gives

$$\frac{a_{ij} - E[S_i]b_j + \sum_k g_k(a_{ijk} - E[S_i]b_{jk})}{a_{jj} - E[S_j]b_j + \sum_k g_k(a_{jjk} - E[S_j]b_{jk})} \tag{10}$$

This is just the fraction of j buyers who would switch to i if j became unavailable. By inspection (11) and (12) are equal, proving the result.

Since equation (10) includes the popular logit, additive, semilog and translog demand models, we conclude that Assumption 2(a) is a not implausible (although not universally true) assumption about consumer behavior.

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Dear Josh,

I enclose four copies of the paper, "Consolidating Distribution Centers Can Reduce Lost Sales", a major revision of "Expected Sales out of Inventory Decrease with Demand Uncertainty" , for publication as a note in **Management Science.**

Sincerely,

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